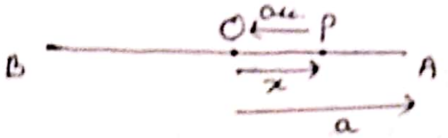
Simple Harmonic Motion and Harmonic oscillator -

A particle is said to execute simple Harmonic motion (S.H.M) if it vibrates to and fro about a fixed point in such a way that at any instant of time restoring force acting on it is proportional to its displacement from a fixed point in its path and is always directed towards that point.

A system executing simple harmonic motion is called harmonic oscillator.



Let us consider a particle executing S. H. M.

mass of particle $= m$

at any time 't', displacement $= x$

$\therefore$ Velocity $= \frac{dx}{dt}$, Acceleration $= \frac{d^2x}{dt}$

Restoring force, $F \propto -x$

$$\therefore m\frac{d^2x}{dt^2} = -cx \quad \text{---} (1)$$

$c \rightarrow$ Constant of proportionality

'$-$'ve sign indicates that force on particle is directed opposite to x increasing. from eqn. (1)

$$m\frac{d^2x}{dt^2} + cx = 0$$

or $$\frac{d^2x}{dt^2} + \frac{c}{m}x = 0 \quad \text{---} (2)$$

or $$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad \text{---} (3)$$

where, $\omega^2 = \frac{c}{m}$

Eqn. (3) is called differential eqn. of motion for a simple harmonic oscillator.

From eqn ① $F = -Cx$ it is clear that harmonic oscillator is an example of conservative system.

At any instant its potential energy

$$U = -\int F\,dx$$

$$\boxed{U = \frac{1}{2} C x^2} \quad \text{—— (4)}$$

Total energy

$$\boxed{E = \frac{1}{2} C x^2 + \frac{1}{2} m v^2} \quad \text{—— (5)}$$

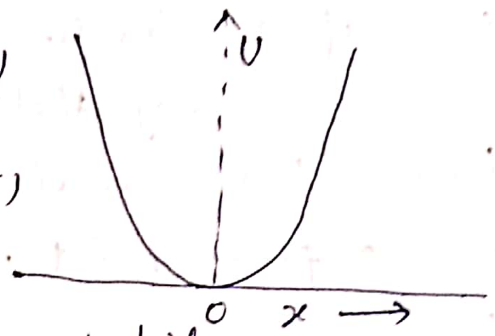


Fig. - Potential energy curve of simple harmonic oscillator.

For a simple harmonic oscillator potential energy curve is a parabola and the force $F = -\frac{dU}{dx}$ acting on it is proportional to displacement in a direction opposite to it.

Now, we will solve eqn (3) by using complex numbers.
The differential eqn of harmonic oscillator is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where} \quad \omega^2 = \frac{C}{m}$$

For this eqn. roots can be find out as

$$(D^2 + \omega^2)x = 0$$

$$\text{or} \quad D^2 = -\omega^2$$

$$D^2 = \sqrt{-\omega^2} = \sqrt{i^2\omega^2}$$

$$D = \pm i\omega$$

Hence, general solution of eqn (3) will be given by

$$x = A e^{i\omega t} + B e^{-i\omega t} \quad \text{—— (6)}$$

where, A & B are constants and can be determined by initial conditions.

From eqn (6) we get

$$x = A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)$$

$$= (A+B)\cos\omega t + (Ai - Bi)\sin\omega t$$

Let,

$$A + B = a \sin\phi$$
$$Ai - Bi = a \cos\phi$$

Then, $x = a \sin\phi \cos\omega t + a \cos\phi \sin\omega t$

$$\boxed{x = a \sin(\omega t + \phi)} \quad \text{——— (7)}$$

Where, a is the maximum value of displacement and is called as amplitude of oscillation and ϕ is called phase const.

Eqn. (7) is the solution of harmonic oscillator and provides us the displacement of particle at any instant of time t.

in eqn. (7) if we put, $\phi = \delta + \pi/2$

then,

$$x = a \sin(\omega t + \delta + \tfrac{\pi}{2})$$
$$\boxed{x = a \cos(\omega t + \delta)} \quad \text{——— (8)}$$

Thus S.H.M. may be represented by either a sine or cosine function, but phase const. has different values.

Velocity → From eqn. (7)

$$x = a \sin(\omega t + \phi)$$

d.w.r. to time we get velocity

$$v = \frac{dx}{dt} = a\omega(\cos\omega t + \phi)$$
$$= a\omega\sqrt{\frac{a^2 - x^2}{a^2}}$$
$$= \frac{\cancel{a}\omega}{\cancel{a}}\sqrt{a^2 - x^2}$$
$$\boxed{v = \omega\sqrt{a^2 - x^2}} \quad \text{——— (9)}$$

from (7)
$\sin(\omega t + \phi) = \frac{x}{a}$
$\therefore \cos(\omega t + \phi) = \sqrt{1 - \frac{x^2}{a^2}}$
$= \sqrt{\frac{a^2 x^2}{a^2}}$

at, mean position, $x = 0$

$$\therefore \boxed{v = a\omega} \text{ (maximum)} \quad \text{—(10)}$$

at, maximum amplitude

$$x = a$$

$$\therefore \boxed{v = 0} \text{ (minimum)} \quad \text{—(11)}$$

Acceleration – from eqn (7)

$$x = a \sin(\omega t + \phi)$$

diff with respect to t

$$\frac{dx}{dt} = a\omega \cos(\omega t + \phi)$$

again differentiating with respect to time 't'

$$\frac{d^2x}{dt^2} = -a\omega^2 \sin(\omega t + \phi)$$

$$= -a\omega^2 \frac{x}{a} \qquad \left[\because \sin(\omega t + \phi) = \frac{x}{a}\right]$$

$$\therefore \text{Acc,} \quad \boxed{\frac{d^2x}{dt^2} = -\omega^2 x} \qquad \text{————— (12)}$$

–ve sign indicates that direction of acceleration is opposite to that of displacement.

at mean position, $x = 0$

$$\therefore \boxed{\frac{d^2x}{dt^2} = 0} \quad (\text{minimum}) \qquad \text{—— (13)}$$

at maximum displacement (amplitude)

$$x = a$$

$$\therefore \boxed{\frac{d^2x}{dt^2} = -a\omega^2} \quad (\text{maximum}) \qquad \text{—— (14)}$$

Physical Significance of constant ω → If time 't' in eqn $x = a \sin(\omega t + \phi)$ is increased by $\frac{2\pi}{\omega}$, the function becomes,

$$x = a \sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$$

$$= a \sin[\omega t + 2\pi + \phi)$$

$$= a \sin(\omega t + \phi)$$

i.e., displacement 'x' remains unchanged. Therefore $\frac{2\pi}{\omega}$ is the time period (T) of S.H.M. of oscillator.

i.e, $\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{c}}}$ $\because \omega^2 = \frac{c}{m}$ — (15)

$\left(\text{Time Period} = \frac{2\pi}{\sqrt{\text{Const}}}\right)$

The number of oscillations per second is called frequency of harmonic oscillator and is given by

$$\boxed{n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{c}{m}}} \quad \text{——— (16)}$$

$\therefore \omega = 2n\pi$

The, quantity ω is called <u>angular frequency</u>.

Examples of Harmonic oscillator

① Simple Pendulam (2) Mass on a spring
③ Torsional Pendulam (4) Helemholtz Resonator
(5) L-C circuit

① <u>Simple Pendulam</u> – A simple pendulam consists of a small, heavy, metallic spherical bob suspended by a thin and flexible and inextensible cotton thread. The upper end of thread is tightly bound with a frictionless rigid support.

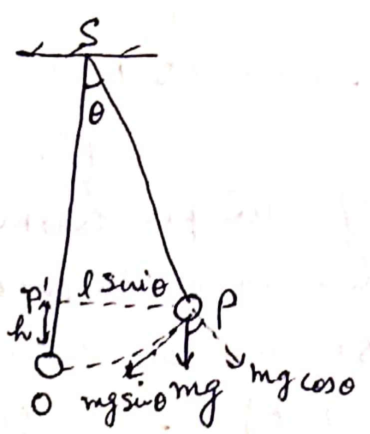


Fig. Simple Pendulam

As the bob is drawn to one side and then left free, it begins oscillate about its mean position 'O' with its natural frequency. Simple harmonic motion starts and the system becomes simple harmonic oscillator.

Let O be the position of equilibrium. S is point of suspension. $\theta \to$ angular displacement at any time 't'
$l \to$ length of inextensible, weightless thread.
$m \to$ mass of bob.

Torque (moment of force) about the point S is

$$\tau = -mgl\sin\theta \quad \text{——} \ (1)$$

Moment of inertia of bob about 'S'

$$I = ml^2 \quad \text{——} \ (2)$$

Torque of bob about S

$$\tau = I\alpha \qquad \alpha \text{- angular acceleration}$$

$$\tau = I\frac{d^2\theta}{dt^2} \quad \text{——} \ (3)$$

from (1) and (3)

$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

$$\cancel{m}l^{\cancel{2}}\frac{d^2\theta}{dt^2} = -\cancel{m}g\cancel{l}\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

or, $$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0} \quad \text{——} \ (4)$$

If θ is small, $\sin\theta = \theta$ $\left(\because \sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} \ldots\right)$

$$\therefore \boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0} \quad \text{——} \ (5)$$ [$\frac{g}{l}$ is const. similar to ω^2]

The eqn (5) is the equation of simple harmonic motion. Hence, simple pendulum is a harmonic oscillator. Solution of eqn. (5) can be given as

$$\theta = \theta_{max}\sin(\omega t + \phi) \quad \text{——} \ (6)$$

where, $\omega = \sqrt{g/l}$ and ϕ is phase constant.

Its, Time period

$$T = \frac{2\pi}{\omega}$$

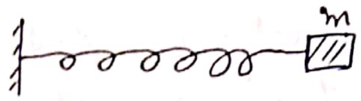
$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}} \quad \text{——— (7)}$$

frequency $n = \frac{1}{\text{Time Period}}$

$$\boxed{n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}} \quad \text{——— (8)}$$

N.B.- In case of simple pendulam the displacement of system is angular hence such a motion is called angular simple harmonic motion.

(2) Mass on a Spring → The system consists of a massless spring. One end of spring is connected to mass 'm' and other end is fixed to a rigid support. The mass and spring



are on a smooth horizontal surface. If a force f is applied to stretch or compress the spring by a displacement x, the mass starts executing simple harmonic motion and the system becomes harmonic oscillator.

Restoring force on spring $F = -cx$ ——— (1)

$c \to$ Force constant.

Also, forces $F = m\frac{d^2x}{dt^2}$ ——— (2)

from (1) & (2) Equation of motion is

$$m\frac{d^2x}{dt^2} = -cx$$

or, $\frac{d^2x}{dt^2} + \frac{c}{m}x = 0$ ——— (3)

eqn. (3) is similar to eqn. of Harmonic oscillator.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{——— (4)}$$

Hence, Spring mass system is a harmonic oscillator.

Time Period, $T = 2\pi\sqrt{\frac{m}{c}}$

frequency, $n = \frac{1}{2\pi}\sqrt{\frac{c}{m}}$

and $x = a \sin(\omega t + \phi)$, where, $\omega = \sqrt{\frac{c}{m}}$

③ <u>Torsional Pendulam</u> → A torsional pendulam consists of thin and long wire. One end of wire is fixed to a rigid support and other end is connected to heavy body (eg- disc or sphere). If the sphere or disc rotated through an angle θ, it executes torsional vibrations of a definite period about the axis of wire.

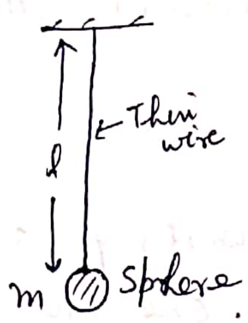


fig - Torsional Pendulam.

$\therefore$ Restoring couple in wire $= -c\theta$ ——— (1)

where, $c = \frac{\eta \pi r^4}{2l}$ is the couple per unit twist of wire.

angular acceleration produced due to this couple, $\alpha = \frac{d^2\theta}{dt^2}$

If I be the moment of inertia of disc or sphere about the axis of wire, then couple is $= I\frac{d^2\theta}{dt^2}$ ——— (2)

from (1) & (2) $\quad I\frac{d^2\theta}{dt^2} = -c\theta$

or. $$\frac{d^2\theta}{dt^2} + \frac{c}{I}\theta = 0 \quad \text{——— (3)}$$

eqn. (3) is similar to equation of harmonic oscillator.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{——— (4)}$$

Hence, Torsional Pendular is a harmonic oscillator.

Time Period, $T = \frac{2\pi}{\omega}$

$= 2\pi\sqrt{\frac{I}{c}}$, Here, $\omega = \sqrt{\frac{c}{I}}$ —(5)

frequency, $n = \frac{1}{2\pi}\sqrt{\frac{c}{I}}$ —— (6)

Solution of eqn (3) is

$\theta = \theta_0 \sin(\omega t + \phi)$ —— (7)

where, $\theta_0 \rightarrow$ Angular amplitude, $\omega = \sqrt{c/I}$, $\phi \rightarrow$ initial Phase.

Inertia table is a modification of torsional pendulam and is employed to determine the moment of inertia of bodies.

(4) Helemholtz Resonator → A Helemholtz resonator is a spherical or cylindrical metallic vessel with narrow neck. The resonator receives the exciting waves through the neck and the air in it acts as a piston which alternately compresses and expands the gas inside the vessel. To detect the resonance a small hole is provided at the other end.

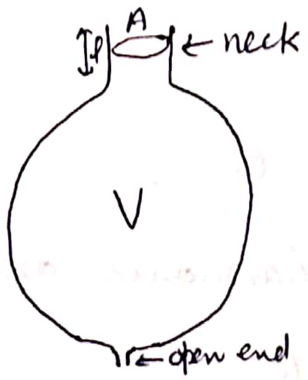


Fig.- Helemholtz resonator

Let, l = length of neck of resonator.

Area of cross section = A

Volume of resonator = V

If the air within the neck moves inwardly a distance x. Then change in volume of gas $\Delta V = xA$ —(1)

Hence, resulting increase in pressure p, may be given by

Bulk Modulus, $K = \frac{\text{Stress}}{\text{Strain}} = \frac{p}{-\Delta V/V} = -\frac{pV}{\Delta V}$

$$\therefore p = -\frac{K.\Delta V}{V} \quad \text{———} \ (2)$$

force exerted due to this pressure

$$F = p.A = -\frac{K\Delta V\, A}{V}$$

$$= -\frac{K.(Ax)\,A}{V}$$

$$F = -\frac{K A^2 x}{V} \quad \text{———} \ (3)$$

Acceleration caused due to force $= \frac{d^2x}{dt^2}$

and mass = density × Volume

$= \rho\, V$

$= \rho\, A l$.

density $= \frac{\text{mass}}{\text{volume}}$

$\rho = \frac{m}{V}$

$m = \rho V$

$= \rho. Al$

$\therefore$ Force = mass × Acc.

$$= \rho A l \frac{d^2x}{dt^2} \quad \text{———} \ (4)$$

from (3) & (4) eqn. of motion is given as

$$\rho A l \frac{d^2x}{dt^2} = -\frac{K A^2 x}{V}$$

$$\text{or,} \quad \frac{d^2x}{dt^2} + \frac{KA}{\rho l V} x = 0 \quad \text{———} \ (5)$$

eqn. (5) is similar to equation of harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{———} \ (6)$$

where, $\omega^2 = \frac{KA}{\rho l V}$

Hence, Helmholtz resonator is a simple harmonic oscillator.

Time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\rho l V}{K}}$

But, $\sqrt{\frac{K}{\rho}} = c$ is velocity of sound in air

$$\therefore T = \frac{2\pi}{c}\sqrt{\frac{lV}{A}} \quad \text{———} \ (7)$$

its frequency of vibration of air of resonator, $n = \frac{c}{2\pi}\sqrt{\frac{A}{V\ell}}$ —— (8)

eqn (8) gives natural frequency of Helmholtz resonator.

Now, if a sound wave is allowed to fall on the open end of resonator then it will resonate in the condition that its natural frequency is equal to the frequency of incident sound wave.

(5) L-C circuit → LC circuit is an example of electrical harmonic oscillator. When a charged capacitor is discharged through a pure inductor then electric oscillations start in the circuit.

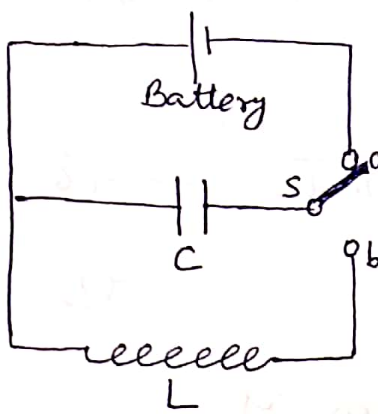


Fig. - L-C circuit

Figure shows an arrangement of L-C circuit. Initially, the condenser C is charged by connecting switch S to a. Now, the switch S is connected to point b and condenser is allowed to discharge through inductor L.

Let, the charge on condenser when charged = q_0

charge on condenser at time t while discharging $= q$

∴ Potential difference between the plates of capacitor at time t, $V = \frac{q}{C}$ —— (1)

Let, the discharging current in inductor L at time t $= i$

∴ induced e.m.f. across the inductor at time t, $e = -L\frac{di}{dt}$ —— (2)

As, there is no resistance in the circuit,

hence, from (1) & (2)

$$\frac{q}{C} = -L\frac{dI}{dt} \quad \text{—— (3)}$$

but, current $I = \frac{dq}{dt}$

$\therefore$ from eqn (3) we get

$$\frac{q}{c} = -L\frac{d^2q}{dt^2}$$

or, $$L\frac{d^2q}{dt^2} + \frac{q}{c} = 0$$

or, $$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 \quad \text{——— (4)}$$

Eqn. (4) is similar to eqn. of Simple Harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{——— (5)}$$

Hence, L-C circuit act as an electric Harmonic oscillator. Here $\omega^2 = \frac{1}{LC}$

Time Period, $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC} \quad \text{——— (6)}.$$

frequency $$f = \frac{1}{2\pi\sqrt{LC}} \quad \text{——— (7)}$$

solution of eqn. (4) may be given as.

$$\boxed{q = q_0 \sin(\omega t + \phi)}, \quad \text{where } \omega = \frac{1}{\sqrt{LC}} \quad \text{— (8)}$$

Thus, we see charge of condenser oscillates between $+q_0$ & $-q_0$, and its frequency depends upon L and C.

Differentiating eqn (8) w. r. to time t, we get instantaneous current, I, i.e,

$$I = \frac{dq}{dt} = q_0\omega \cos(\omega t + \phi)$$

$$\boxed{I = I_0 \cos(\omega t + \phi)} \quad (I_0 = q_0\omega) \quad \text{——— (9)}$$

N.B. - Practically, a LC circuit has always some resistance. In present analysis we have considered pure inductor (zero resistivity

Ques - What do you mean by damped oscillations? obtain a differential equation of damped harmonic oscillator and solve it. Discuss over damped (high damping), Under damped (Low damping) and critically damped cases.

Answer - Damped Oscillations ~~or~~ Damped Harmonic oscillator

In actual practice, the amplitude of oscillations of a Harmonic oscillator gradually decreases to zero due to frictional forces of medium (viscosity etc.) in which oscillator is moving. Such oscillations are called damped oscillations and oscillator is called as damped harmonic oscillator.

A damped harmonic oscillator experiences following forces.

(i) a restoring force, $-cx$

(ii) a damping force, $-\gamma \frac{dx}{dt}$ $\left(\frac{dx}{dt} = v \text{ is small}\right)$

damping force is proportional to velocity. γ is a constant.

Hence, eqn. of motion of damped harmonic oscillator is

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - cx$$

or, $$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{c}{m}x = 0$$

or, $$\boxed{\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = 0} \quad \text{——— (1)}$$

where, $2k = \frac{\gamma}{m} = \frac{1}{\tau}$, τ is relaxation time, γ is damping const.

$\omega_0 = \sqrt{\frac{c}{m}}$ is natural angular frequency in absence of damping forces

Eqn. (1) is differential equation of damped harmonic oscillator.

differential eqn. of damped harmonic oscillator can be solved as -

$$\alpha^2 x + 2K\alpha x + \omega_0^2 x = 0$$

or, $(\alpha^2 + 2K\alpha + \omega_0^2)x = 0$

or, $$\alpha = \frac{-2K \pm \sqrt{4K^2 - 4\omega_0^2}}{2}$$

$$\alpha = -K \pm \sqrt{K^2 - \omega_0^2}$$

$\therefore$ Solutions of eqn ① are $\alpha_1 = (-K + \sqrt{K^2-\omega_0^2})$ and $\alpha_2 = (-K - \sqrt{K^2-\omega_0^2})$

$$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

$$= Ae^{(-K+\sqrt{K^2-\omega_0^2})t} + Be^{(-K-\sqrt{K^2-\omega_0^2})t}$$

$$x = e^{-Kt}\left(Ae^{\sqrt{K^2-\omega_0^2}\,.t} + Be^{-\sqrt{K^2-\omega_0^2}\,.t}\right) \quad \text{——— (2)}$$

where, A and B are constants, depending upon initial conditions of motion.

Now, depending upon the value of K & ω_0, the factor $\sqrt{K^2-\omega_0^2}$ may be imaginary, real or zero.

<u>Case i</u> - <u>Underdamped case or Low damping</u> -

if $K < \omega_0$, the damping is low or underdamping.

then, $\sqrt{K^2-\omega_0^2} = \sqrt{-1(\omega_0^2 - K^2)}$

$$= i\sqrt{\omega_0^2 - K^2} \qquad (\because i^2 = -1)$$

$$= i\omega \qquad \text{Here } \omega = \sqrt{\omega_0^2 - K^2} \text{ is real}$$

Now, from eqn (2) we have,

$$x = e^{-Kt}\left(Ae^{i\omega t} + Be^{-i\omega t}\right)$$

$$= e^{-Kt}\left[A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)\right]$$

$$= e^{-Kt}\left[(A+B)\cos\omega t + i(A-B)\sin\omega t\right] \quad \text{—③}$$

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As x is real quantity, $(A+B)$ and $(A-B)$ must be real quantities.

If $A+B = a_0 \sin\phi$

and $i(A-B) = a_0 \cos\phi$

Then, from eqn. (3)

$$x = e^{-kt}[a_0 \sin\phi \cos\omega t + a_0 \cos\phi \sin\omega t]$$

$$\boxed{x = a_0 e^{-kt} \sin(\omega t+\phi)} = a \sin(\omega t+\phi) \quad \text{--- (4)}$$

Here, $a = a_0 e^{-kt}$ is amplitude of damped harmonic oscillator.

Eqn. (4) represents law damped harmonic oscillator motion. This motion is oscillatory or ballistic.

Time Period, $T = \frac{2\pi}{\omega}$

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - k^2}} \quad \text{--- (5)}$$

freq. $$n = \frac{1}{T} = \frac{\sqrt{\omega_0^2 - k^2}}{2\pi} \quad \text{--- (6)}$$

Thus, the effect of damping is to increase the periodic time.

If there is no damping i.e. $k = 0$

then, $$T = \frac{2\pi}{\omega_0} \quad \text{--- (7)}$$

The amplitude of damped motion $(a_0 e^{-kt})$ is represented as in fig. below:-

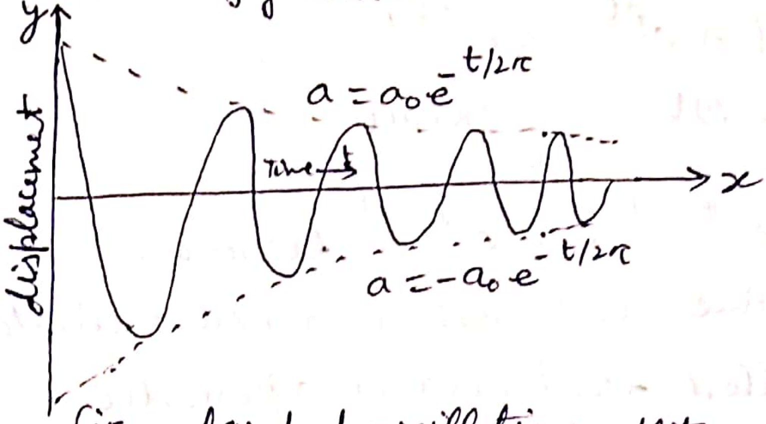

fig. - damped oscillations amplitude (Time displacement curve)

$a_0 \rightarrow$ amplitude in absence of damping.

fig. shows that in presence of damping amplitude decreases exponentially

$$a = a_0 e^{-kt} \quad \text{when } \sin(\omega t+\phi) = +1 \quad \text{--- (8)}$$

and $$a = -a_0 e^{-kt} \quad \text{when } \sin(\omega t+\phi) = -1 \quad \text{--- (9)}$$

The time interval between successive maximum displacements (i.e. amplitudes) is $T/2$, hence if a_n and a_{n+1} are successive amplitudes, then

$$a_n = a_0 e^{-Kt}$$

and $$a_{n+1} = a_0 e^{-K(t+\frac{T}{2})}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{a_0 e^{-Kt}}{a_0 e^{-K(t+\frac{T}{2})}}$$

$$\frac{a_n}{a_{n+1}} = e^{KT/2} = d \text{ (a const. for motion)} \quad \text{—— (10)}$$

Here, d is called the decrement, indication for the reduction in amplitude.

Now, from (10) we get,

$$\log_e d = \frac{KT}{2} = \lambda \quad \text{—— (11)}$$

The quantity λ is called the logarithmic decrement.

Ques:- Discuss logarithmic decrement in damped harmonic oscillator.

Case ii - Overdamped case or High Damping →

If, $K > \omega_0$, the damping is high or over damping.

Then $\sqrt{K^2-\omega_0^2}$ is real quantity.

Let, $\sqrt{K^2-\beta^2} = \beta$

Then from eqn (2) we get.

$$x = e^{-Kt}(Ae^{\beta t} + Be^{-\beta t})$$

$$x = Ae^{-(K-\beta)t} + Be^{-(K+\beta)t} \quad \text{—— (12)}$$

as, $K > \beta$, hence $e^{-(K-\beta)}$ and $e^{-(K+\beta)}$ decreases exponentially with time and motion is non oscillatory. Such motion is called dead beat or aperiodic.

eg.- dead beat galvanometer.

Case iii - Criticalled damped case →

If, $K = \omega_0$

Than $\sqrt{K^2-\omega_0^2} = 0$, than from eqn (2) we get

$$x = (A+B)\cdot e^{-kt}$$

$$x = c\cdot e^{-kt} \quad \text{where, } c = A+B \quad \text{——— (13)}$$

In eqn (13) there is only one constant, hence it does not provide us the solution of eqn (1).

Now, suppose, $\sqrt{k^2-\omega_0^2} = h$, h is a small quantity.

Hence, from eqn (2) we get

$$x = e^{-kt}\left(A e^{ht} + B e^{-ht}\right)$$

$$= e^{-kt}\left[A\left(1+ht+\frac{h^2t^2}{\lfloor 2}+\cdots\right)+B\left(1-ht+\frac{h^2t^2}{\lfloor 2}\cdots\right)\right]$$

h is small, hence h^2 is very small, hence neglecting higher termes, we get,

$$x = e^{-kt}\left[(A+B)+h(A-B)t\right]$$

$$x = e^{-kt}\left[P+Qt\right] \quad \text{——— (14)}$$

where, $P = A+B$ and $Q = h(A-B)$

differentiating eqn (14) with respect to time t

$$\frac{dx}{dt} = e^{-kt}\cdot Q + (P+Qt)\,e^{-kt}(-k) \quad \text{——— (15)}$$

initially, at $t=0$, $x = x_0$, and $v = v_0$

then, $x_0 = P$ (from 14) ——— (16)

and $v_0 = Q - kP$ (from 15) ——— (17)

$\therefore Q = v_0 + kP = v_0 + kx_0$

$\therefore$ eqn (14) can be written as using (16) & (17)

$$x = e^{-kt}\left[x_0 + (v_0 + kx_0)t\right] \quad \text{——— (18)}$$

eqn (17) represents that initially displacement increases due to the factor $[x_0 + (v_0+kx_0)t]$. But later on with increase in time the exponential term becomes relatively more important and displacement becomes zero and oscillatory motion does not occur. This is called critically damped or just aperiodic.

Ques. - Discuss Energy dissipation or Power dissipation.

Sol - Power Dissipation or Rate of Dissipation of Energy →

If a particle oscillates in a medium, then due to viscosity of medium damping forces act on the particle in a direction opposite to its movement. In this process work is done by the particle in overcoming the resistance forces. Consequently, the mechanical energy of the vibrating particle continuously decreases so that the amplitude of oscillation becomes less and less. Here we will find the relation for rate of dissipation of energy i.e. Power dissipation.
At any instant of time t; the displacement of a damped harmonic oscillator is given by

$$x = a_0 e^{-kt} \sin(\omega t + \phi) \quad \text{—— (1)}$$

∴ Velocity of particle $v = \frac{dx}{dt}$

$$= a_0 e^{-kt} \cdot \omega \cos(\omega t + \phi) - k a_0 e^{-kt} \sin(\omega t + \phi)$$

$$= a_0 e^{-kt} [-k \sin(\omega t + \phi) + \omega \cos(\omega t + \phi)]$$

∴ Kinetic energy of vibration, $KE = \frac{1}{2} m v^2$

$$K.E. = \frac{1}{2} m a_0^2 e^{-2kt} [k^2 \sin^2(\omega t + \phi) + \omega^2 \cos^2(\omega t + \phi) - 2k\omega \sin(\omega t + \phi) \cos(\omega t + \phi)] \quad \text{—— (2)}$$

Potential energy,

$$P.E. = \frac{1}{2} C x^2 = \frac{1}{2} m \omega_0^2 x^2 \qquad \left(P.E. = \int_0^x Cx\,dx\right)$$

$$= \frac{1}{2} m a_0^2 e^{-2kt} [\omega_0^2 \sin^2(\omega t + \phi)] \quad \text{—— (3)}$$

Using (1)

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The average total energy for a period will be the sum of time average of kinetic and potential energy. If amplitude of oscillation does not change much, then the factor e^{-2kt} may be taken as constant. The average values of $\sin^2(\omega t+\phi)$, $\cos^2(\omega t+\phi)$ and $2\sin(\omega t+\phi)\cos(\omega t+\phi)$ for a period are $\frac{1}{2}$, $\frac{1}{2}$ and 0 respectively.

Hence, average kinetic energy

$$= \frac{1}{2} m a_0^2 e^{-2kt}\left[k^2 \cdot \frac{1}{2} + \omega^2 \cdot \frac{1}{2}\right]$$

$$= \frac{1}{4} m a_0^2 \omega^2 e^{-2kt}$$ [as damping is low so $k^2 << \omega^2$) —— (4)

and average potential energy

$$= \frac{1}{4} m a_0^2 \omega^2 e^{-2kt}$$ —— (5)

∴ Average total energy

$$\boxed{E = \frac{1}{2} m a_0^2 \omega^2 e^{-2kt}}$$ i.e. $E \propto a_0^2$ and $E \propto \omega^2$

$$E = E_0 e^{-2kt} \text{ or } E_0 e^{-t/\tau}$$ (as $2k = \frac{1}{\tau}$) —— (6)

∴ average Power dissipation

$$P = -\frac{dE}{dt}$$

$$= k m a_0^2 \omega^2 e^{-2kt}$$

$$\boxed{P = 2kE \text{ or } \frac{E}{\tau}}$$ —— (7)

This dissipation of energy is caused by the damping force $\gamma \frac{dx}{dt}$. This loss of energy generally appears in the form of heat in oscillating system.

Ques - What is meant by quality factor of an oscillator? How is it affected by damping?

Ans. - Quality Factor Q → quality factor ~~of an oscillating~~ ~~system~~ represents the efficiency of an oscillating system.
Quality factor of as oscillator is defined as 2π times the ratio of energy stored to the average energy loss per period.

$$Q = 2\pi \frac{\text{Energy stored}}{\text{Energy loss per period}}$$

$$= 2\pi \frac{E}{PT} \qquad \left(\because \frac{2\pi}{T} = \omega\right)$$

$$= \frac{E\omega}{P}$$

$$\text{Or, } \boxed{Q = \omega\tau} \quad (1) \quad \left(\because P = \frac{E}{\tau} \text{ or } \tau = \frac{E}{P}\right)$$

Q is a dimensionless quantity.
In case of low damping $Q = \omega_0 \tau$
But, $\omega_0 = \sqrt{c/m}$ and $\tau = \frac{m}{\gamma}$

$$\therefore \boxed{Q = \sqrt{\frac{mc}{\gamma}}} \quad (2)$$

Thus high value of Q means that the damping of the oscillating system is low. This Q is a measure of the extent to which oscillator is free from damping. For an undamped oscillator $\gamma = 0$, so that Q is infinite.

Ques: Give the theory of Driven or forced Harmonic oscillators.

Ans. Driven or forced Harmonic oscillator →

If an external periodic force is applied on a damped harmonic oscillator, then the oscillating system is called driven or forced harmonic oscillator and its oscillations are called forced or driven oscillations.

Let the periodic force on oscillating particle of mass m be,

$$F = F_0 \sin pt$$

where, F_0 – Maximum value of force of freq. $\frac{p}{2\pi}$

Let, x be the displacement from equilibrium at time t, the velocity be $\frac{dx}{dt}$. The forces acting upon the system are

① A restoring force $(-Cx)$

② A frictional force $(-\gamma \frac{dx}{dt})$ or damping force. γ is damping constant.

③ External periodic force $F = F_0 \sin pt$, p is angular freq.

Thus, total force acting on system

$$F = -Cx - \gamma \frac{dx}{dt} + F_0 \sin pt$$

Hence, differential eqn. of motion is

$$m\frac{d^2x}{dt^2} = -\gamma \frac{dx}{dt} - Cx + F_0 \sin pt$$

or, $$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{C}{m}x = \frac{F_0}{m}\sin pt$$

or, $$\boxed{\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt} \quad \text{—(1)}$$

where, $2K = \frac{\gamma}{m}$, $\omega_0 = \sqrt{C/m} = 2\pi n_0$ = natural angular freq.

and $f_0 = \frac{F_0}{m}$

The natural freq. of oscillating system may be different to freq. $p/2\pi$ of applied force.

The complete solution of eqn (1) will consist of sum of Complementary function and particular integral.

Homogeneous part of equation is –

$$(D^2 + 2KD + \omega_0^2)x = 0$$

$$\therefore D = -K \pm \sqrt{K^2 - \omega_0^2}$$

and the complementary function is

$$CF = e^{-Kt}\left[A e^{\sqrt{K^2-\omega_0^2}\,t} + B e^{-\sqrt{K^2-\omega_0^2}\,t}\right]$$

$$= e^{-Kt}\left[A e^{i\sqrt{\omega_0^2-K^2}\,t} + B e^{-i\sqrt{\omega_0^2-K^2}\,t}\right] \quad (K < \omega_0)$$

$$= e^{-Kt}\left[A\left(\cos\sqrt{\omega_0^2-K^2}\,t + i\sin\sqrt{\omega_0^2-K^2}\,t\right) + B\left(\cos\sqrt{\omega_0^2-K^2}\,t - i\sin\sqrt{\omega_0^2-K^2}\,t\right)\right]$$

$$= e^{-Kt}\left[(A+B)\cos\sqrt{\omega_0^2-K^2}\,t + i(A-B)\sin\sqrt{\omega_0^2-K^2}\,t\right]$$

$$CF = A_0 e^{-Kt}\sin\left(\sqrt{\omega_0^2-K^2}\,t + \phi\right) \quad \text{——— (2)}$$

where, A_0 & ϕ are new constants.

The Particular integral is

$$PI = \frac{1}{(D^2 + 2KD + \omega_0^2)} f_0 \sin pt$$

$$= A\sin(pt - \theta) \quad \text{——— (3)}$$

where, A is amplitude & θ is phase of oscillations. Here, θ gives the phase difference of oscillations and driving force.

General solution of forced or driven oscillator is given as

$$x = CF + PI$$

$$= A_0 e^{-Kt}\sin\left(\sqrt{\omega_0^2-K^2}\,t + \phi\right) + A\sin(pt - \theta) \quad \text{——— (4)}$$

The first term is called as transient term and vanishes away with time as e^{-Kt}. During transient state the oscillator oscillates neither with the its natural frequency nor with the frequency of impressed force.

The second term is called as steady state term and governs the motion of oscillator. During the steady state the oscillator performs forced or driven oscillations with the frequency of impressed force.

Ques. - Discuss steady state motion of forced damped harmonic oscillator.

Ans. - The equation of motion of forced or driven damped harmonic oscillator is given as

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt \quad \text{——} (1)$$

where, $2K = \frac{\gamma}{m}$, $\omega_0 = \sqrt{c/m}$

The steady state solution of eqn (1) is given as -

$$x = A \sin(pt - \theta) \quad \text{——} (2)$$

Where, $A \rightarrow$ amplitude of forced oscillations. Frequency of vibrations is $p/2\pi$, and θ represents phase difference between force and resultant displacement of system.

From eqn (2) we get

$$\frac{dx}{dt} = pA\cos(pt-\theta) \quad \text{——} (3)$$

and, $$\frac{d^2x}{dt^2} = -p^2 A \sin(pt-\theta) \quad \text{——} (4)$$

Using (3) & (4), eqn (1) gives

$$-p^2 A \sin(pt-\theta) + 2KpA\cos(pt-\theta) + \omega_0^2 A \sin(pt-\theta) = f_0 \sin\{(pt-\theta)+\theta\}.$$

or. $$A(\omega_0^2 - p^2)\sin(pt-\theta) + 2KpA\cos(pt-\theta) = f_0 \sin(pt-\theta)\cos\theta + f_0\cos(pt-\theta)\sin\theta \quad \text{——} (5)$$

This eqn (5) holds for all values of t. Hence coefficients of $\sin(pt-\theta)$ and $\cos(pt-\theta)$ at L.H.S. and R.H.S. must be equal.

$$\therefore \quad A(\omega_0^2 - p^2) = f_0 \cos\theta \quad \text{—— (6)}$$

$$2kpA = f_0 \sin\theta \quad \text{—— (7)}$$

squaring and adding eqns (6) & (7) we get

$$A^2(\omega_0^2-p^2)^2 + 4k^2p^2A^2 = f_0^2(\cos^2\theta + \sin^2\theta)$$

$$A^2\left[(\omega_0^2-p^2)^2 + 4k^2p^2\right] = f_0^2$$

$$\therefore \quad \boxed{A = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2 + 4k^2p^2}}} \quad \text{—— (8)}$$

dividing eqn.(7) by (6) we get,

$$\tan\theta = \frac{2kp\cancel{A}}{\cancel{A}(\omega_0^2-p^2)}$$

$$\text{or,} \quad \boxed{\theta = \tan^{-1}\left(\frac{2kp}{\omega_0^2-p^2}\right)} \quad \text{—— (9)}$$

substituting the values of A and θ in eqn (2) we get solution of steady state equation of motion as

$$x = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2+4k^2p^2}} \sin\left\{pt - \tan^{-1}\left(\frac{2kp}{\omega_0^2-p^2}\right)\right\} \quad \text{—— (10)}$$

Ques - Discuss mechanical impedance of forced oscillar.

Ans. - Steady state solution of forced damped harmonic oscillator is

$$x = A\sin(pt-\theta), \text{ where, } A = \frac{f_0}{\sqrt{(\omega_0^2-p^2)^2+4k^2p^2}} \quad \text{—— (1)}$$

and, $f_0 = \frac{F_0}{m}$ is amplitude of driving force.

$$\therefore A = \frac{F_0}{\sqrt{(m\omega_0^2 - mp^2)^2 + 4m^2k^2p^2}}$$

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$$A = \frac{F_0}{p\sqrt{\left(\frac{m\omega_0^2}{p} - mp\right)^2 + (2mk)^2}} \quad \text{and } \omega_0^2 = \frac{c}{m}$$

$$A = \frac{F_0}{p\sqrt{\left(mp - \frac{c}{p}\right)^2 + (2mk)^2}}$$

Let, $2mk = R_m$, R_m is defined as mechanical resistance.

$$\therefore A = \frac{F_0}{p\sqrt{R_m^2 + \left(mp - \frac{c}{p}\right)^2}}$$

$$A = \frac{F_0}{p Z_m} \quad \text{—— (2)}$$

Where, $$Z_m = \sqrt{R_m^2 + \left(mp - \frac{c}{p}\right)^2} \quad \text{—— (3)}$$

Z_m is called as mechanical impendance of oscillator from the analogy of electric impendance

Mechanical Impedance is defined as the force required to produce unit velocity in the oscillator.

$\therefore$ eqn. ① can be written as—

$$\boxed{x = \frac{F_0}{p Z_m} \sin(pt - \theta)} \quad \text{—— (4)}$$

Ques. → Explain Dependence of amplitude of forced harmonic oscillator on the frequency of applied force. Show that at resonance the displacement always lags behind the driving force. Discuss sharpness of resonance.

Ans. → We know that differential eqn of motion of forced or driven harmonic oscillator is given as

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt \quad \text{—— ①}$$

steady state solution of eqn (1) is given as

$$x = A \sin(pt - \theta) \quad \text{--- (2)}$$

where,
$$A = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} \quad \text{--- (3)}$$

and.
$$\tan\theta = \frac{2kp}{\omega_0^2 - p^2} \text{ or } \theta = \tan^{-1}\left(\frac{2kp}{\omega_0^2 - p^2}\right) \quad \text{--- (4)}$$

and, $2k = \frac{r}{m}$, $\omega_0 = \sqrt{C/m} = 2\pi n_0$ = natural angular freq. of system in absence of damping and driven force.

$f_0 = \frac{F_0}{m}$, F_0 is maximum value of force of frequency $\frac{p}{2\pi}$.

In steady state the phase depends on the relative magnitudes of the driving and natural frequencies p & ω_0 respectively.

Here, we will assume damping is low. As such three cases arise (i) $\omega_0 >> p$, $\omega_0 = p$ and $\omega_0 << p$.

Case (i) Low driving frequency -

When $\omega_0 >> p$, $\tan\theta \to 0$ or $\theta \to 0$

Hence, driving force and displacement are in same phase. Hence amplitude of oscillations from eqn (3) is given by

$$A = \frac{f_0}{\omega_0^2} = \frac{F_0/m}{C/m}$$

$$A = \frac{F_0}{C} \quad \text{--- (5)}$$

Thus in this case amplitude does not depend upon mass but force constant.

Now, if driving frequency is increased, the value of amplitude A increases. Eqn (3) can be arranged as

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - 2k^2 - p^2)^2 + 4k^2\omega_0^2 - 4k^4}} \quad \text{--- (6)}$$

Therefore, to have maximum value of A.

$$\omega_0^2 - 2k^2 - p^2 = 0$$

or, $p = \sqrt{\omega_0^2 - 2k^2} = p_r$ (say) — (7)

$$\therefore \boxed{A_{max} = \frac{f_0}{2k\sqrt{\omega_0^2 - k^2}}} = \frac{f_0 \tau}{\sqrt{\omega_0^2 - k^2}} \quad \text{— (8)} \quad \text{from (6)}$$

This phenomenon in which the amplitude of the driven or forced oscillator becomes maximum at a particular driven frequency, is <u>amplitude resonance</u>, and this freq. is known as resonante frequency.

If the damping is low

then $\omega_0 = p_r$ $\quad\quad$ $p_r \rightarrow$ driven freq.

and amplitude,

$$A_{max} = \frac{f_0}{2k\omega_0} \text{ or } \frac{f_0 \tau}{\omega_0} \quad \text{— (9)}$$

Hence, in case of zero damping amplitude should be <u>infinity</u>.

If the damping is low, then ratio of amplitude at resonance to the amplitude at zero driven frequency is given by

$$= \frac{f/2k\omega_0}{f/\omega_0^2} = \frac{\omega_0}{2k}$$

$$= \omega_0 \tau = Q \quad \text{—— (10)}$$

Q is quality factor.

Thus we see damping controls amplitude of resonance.

<u>Case ii</u> – When $p = \omega_0$

from eqn (3) and (4) we get

$$\tan\theta = \infty \text{ or } \theta = \frac{\pi}{2} \quad \text{—— (11)}$$

and $A = \dfrac{f_0}{2k\omega_0}$

$$A = \frac{f_0 \tau}{\omega_0} \quad \text{—— (12)}$$

which is less than amplitude as given by eqn (8). In case of low damping, the resonant frequency is equal to the natural frequency of oscillator and the amplitude is maximum which is known as resonance.

Case iii - High driving frequency when $\omega_0 << p$.

In this case.

$$\tan\theta = -\frac{2k}{p} \qquad \text{from (4)}$$

$$\theta \rightarrow -0$$

$$\text{or } \theta \rightarrow \pi \qquad \text{—— (13)}$$

and amplitude of resulting oscillation

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}}$$

becomes,

$$A = \frac{f_0}{\sqrt{p^4 + 4k^2p^2}} \simeq \frac{f_0}{p^2} \text{ (nearly)}$$

$$A \simeq \frac{f_0}{p^2} = \frac{F_0}{m p^2} \qquad \text{—— (14)}$$

So, on increasing the driving frequency (p) the amplitude decreases and phase tends to π.

The variation of amplitude A with the frequency p of the periodic force is shown in fig. 1. Initially amplitude A continuously increases with increase in angular freq. p of force and reaches its maximum value at ω_0. This is the condition of amplitude resonance. Now on further increasing p, the amplitude decreases gradually. For low damping the height of peaks become greater. When damping is zero height of peak become infinite.

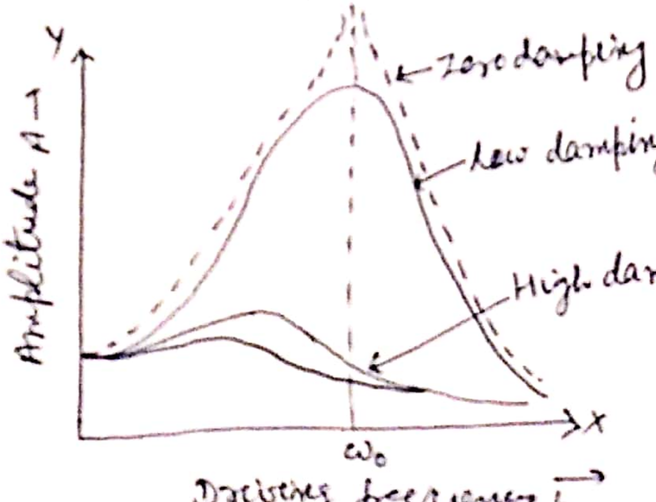
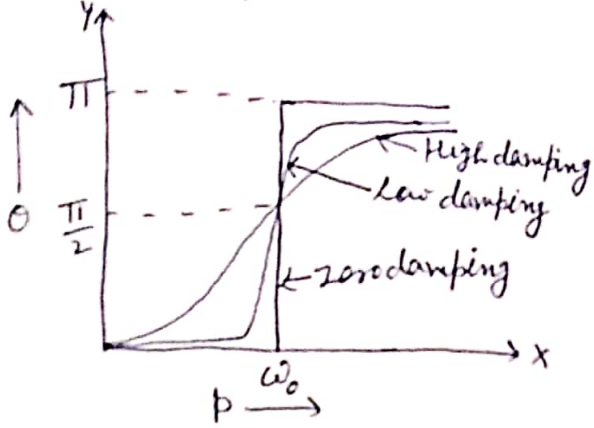


Fig. 1

The variation of phase difference θ with the frequency p is shown in fig 2. From the above discussion it is also clear that value of θ increases from 0 to π and it always remains positive. This means that resulting displacement of oscillator always lags in phase from the driven force.



Ques. - Discuss Sharpness of resonance.

Ans. - Sharpness of resonance means the rate of fall in amplitude with the change of forcing frequency on each side of resonant frequency. In fig 1 different values of damping curves have been drawn. When damping is low, the ~~resonance~~ amplitude falls of very rapidly and we say resonance is sharp. eg. - Resonance of an sonometer wire with a tuning fork.

On the other hand, for high damping the ~~resonance~~ amplitude falls of very slowly on ~~eg~~ either side of resonant frequency and resonance is said to be flat. eg. - Resonance of an air column with a sonometer.

Ques. - Discuss Velocity resonance.

Ans. The solution of steady state motion of forced damped harmonic oscillator is given as

$$x = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} \sin(pt - \theta) \qquad \text{—(1)}$$

∴ The velocity v of forced oscillator is

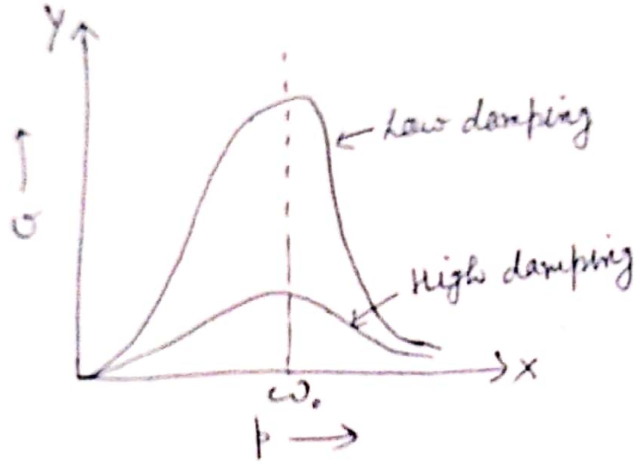
$$v = \frac{dx}{dt} = \frac{f_0\, p}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} \cos(pt - \theta) \qquad \text{—(2)}$$

$$v = v_0 \sin\left(pt - \theta + \frac{\pi}{2}\right) \qquad \text{—(3)}$$

where, $v_0 = \dfrac{f_0 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}}$ is amplitude of velocity.

$$v_0 = \frac{f_0}{\sqrt{\frac{(\omega_0^2 - p^2)^2}{p^2} + 4k^2}} \quad \text{—— (4)}$$

and $$\theta = \tan^{-1}\left(\frac{2kp}{\omega_0^2 - p^2}\right) \quad \text{—— (5)}$$



from eqn (4) we see that that velocity amplitude varies with the driven frequency as shown in figure.
When $p = 0$, $v_0 = 0$ and
when $p = \omega_0$, v_0 is maximum.
Hence, at the frequency of driven force $p = \omega_0 = \sqrt{\frac{c}{m}}$ the velocity has the maximum value and we call it <u>velocity resonance</u>.

at velocity resonance, $\tan\theta = \dfrac{2kp}{\omega_0^2 - p^2} = \dfrac{2k\omega_0}{\omega_0^2 - \omega_0^2}$

$$= \infty$$

or $$\boxed{\theta = \pi/2} \quad \text{—— (6)}$$

$\therefore$ Velocity phase const. $= -\theta + \frac{\pi}{2} = -\frac{\pi}{2} + \frac{\pi}{2} = 0$

This means that velocity of the oscillator is in phase with the applied force.

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Ques – Show that at the frequency of velocity resonance, the power absorption in a driven or forced oscillator is maximum.

or

Discuss power absorption in a forced harmonic oscillator or Discuss power supplied by driving force to the forced oscillator.

Ans – Power Absorption → When an oscillator executes vibrations in presence of damping forces, it loses energy in doing work against these forces. This loss of energy is supplied by the periodic impressed force so as to continue the oscillations. Energy supplied by driving force ($F = F_0 \sin pt$) will be equal to work done.

$$dE = F\,dx = F \cdot \frac{dx}{dt}\,dt$$

or $$\frac{dE}{dt} = F\frac{dx}{dt} \qquad \text{(dx is displacement in time dt)} \quad \text{—(1)}$$

We know,

$$F = F_0 \sin pt$$

$$F = m f_0 \sin pt \qquad \text{—— (2)}$$

and from steady state solution

$$x = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} \sin(pt - \theta)$$

$$\therefore \frac{dx}{dt} = \frac{f_0 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} \cos(pt - \theta) \qquad \text{—— (3)}$$

hence, any time energy absorbed by the oscillating system is

$$p = \frac{dE}{dt} = F\frac{dx}{dt}$$

$$= m f_0 \sin pt \cdot \frac{f_0 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} \cos(pt - \theta)$$

$$= \frac{m f_0^2 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} \sin pt \cdot \cos(pt - \theta) \qquad \text{—— (4)}$$

Hence, the average power absorbed is

$$P_{av} = \left(F \frac{dx}{dt} \right)_{av}$$

$$P_{av} = \frac{m f_0^2 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4K^2 p^2}} \left[\sin pt \cos(pt - \theta) \right]_{av} \quad \text{— (5)}$$

Now, the average of $\sin pt \cos(pt-\theta)$ for one period T or $2\pi/\omega$

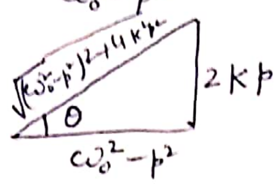
$$= \frac{1}{T} \int_0^T \sin pt \cos(pt - \theta)\, dt$$

$$= \frac{1}{T} \int_0^T \frac{1}{2} \{ \sin(2pt - \theta) - \sin\theta \}\, dt$$

$$= \frac{\sin\theta}{2}$$

$$= \frac{1}{2} \cdot \frac{2Kp}{\sqrt{(\omega_0^2 - p^2)^2 + 4K^2 p^2}}$$

$$\tan\theta = \frac{2Kp}{\omega_0^2 - p^2}$$



from eqn (5)

$\therefore$

$$\boxed{P_{av} = \frac{1}{2} m f_0^2 \frac{2Kp^2}{(\omega_0^2 - p^2)^2 + 4K^2 p^2}} \quad \text{—(6)}$$

or $$P_{av} = mK \frac{f_0^2 p^2}{(\omega_0^2 - p^2)^2 + 4K^2 p^2}$$

$$= mK v_0^2 \qquad \text{where, } v_0^2 = \frac{f_0^2 p^2}{(\omega_0^2 - p^2)^2 + 4K^2 p^2}$$

$$\boxed{P_{av} = \frac{m v_0^2}{2\tau} \text{ or } \gamma v_0^2} \quad \text{——— (7)}$$

From eqn. (6), when $p = \omega_0$

we get absorbed average power is maximum and is equal to $\frac{1}{4} m f_0^2 / K$ or $\frac{1}{2} m f_0^2 \tau$.

Thus the power absorbed is maximum at the frequency of velocity resonance and not at the frequency of amplitude resonance.

Ques: - Give the theory of oscillations in a LCR electrical circuit.

Sol. Driven LCR circuit - In a driven LCR circuit or harmonic oscillator the oscillations are sustained by an alternating electromotive force.

Let's consider a circuit containing Resistance R, inductance L and capacitance C in series.

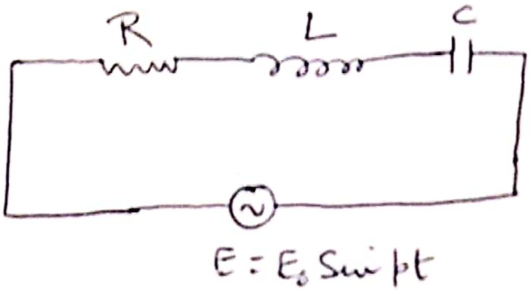


Fig. - LCR circuit

Alternating E.m.f. applied

$$E = E_0 \sin pt \quad \text{--- (1)}$$

at any instant 't' potential difference across the plates of condenser C.

$$V_C = \frac{Q}{C} \quad \text{--- (2)}$$

If at this instant of time 't' the current in the circuit is I then self induced e.m.f. in the inductance,

$$V_L = -L\frac{dI}{dt} \quad \text{--- (3)}$$

According to ohm's Law potential difference across resister

$$V_R = IR \quad \text{--- (4)}$$

Hence, total Potential diff. = total E.M.F.

$$IR + \frac{Q}{C} = E - L\frac{dI}{dt} \quad \text{--- (5)}$$

or $$E = IR + \frac{Q}{C} + L\frac{dI}{dt}$$

or $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0 \sin pt \qquad I = \frac{dQ}{dt}$$

or $$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{E_0}{L}\sin pt \quad \text{--- (6)}$$

this eqn(6) is similar to eqn of driven harmonic oscillator

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt \quad \text{--- (7)}$$

where, $2k = \frac{R}{L}$, $\omega_0^2 = \frac{1}{LC}$, $f_0 = \frac{E_0}{L}$ ——— (8)

Hence, oscillations of LCR circuit are driven harmonic oscillations and circuit acts as driven harmonic oscillator.

$\therefore$ Steady state solution of eqn (6) can be given as

$$q = \frac{E_0/L}{\sqrt{\left(\frac{1}{LC} - p^2\right)^2 + \left(\frac{pR}{L}\right)^2}} \sin(pt - \theta) \qquad \text{——— (9)}$$

and $\tan\theta = \dfrac{pR/L}{\left(\frac{1}{LC} - p^2\right)}$ or $\theta = \tan^{-1}\left(\dfrac{pR/L}{\frac{1}{LC} - p^2}\right)$ — (10)

differentiating eqn. (9) w.r.t time 't' we get the current $\left(I = \frac{dq}{dt}\right)$ in the circuit.

$$\therefore I = \frac{dq}{dt} = \frac{E_0 p}{L\sqrt{\left(\frac{1}{LC} - p^2\right)^2 + \left(\frac{pR}{L}\right)^2}} \cos(pt - \theta)$$

$$= \frac{E_0}{\frac{L}{p}\sqrt{\left(\frac{1}{LC} - p^2\right)^2 + \left(\frac{pR}{L}\right)^2}} \cos(pt - \theta)$$

$$I = \frac{E_0}{\sqrt{R^2 + \left(pL - \frac{1}{pC}\right)^2}} \sin(pt - \phi) \qquad \left[\text{where } \phi = \theta - \frac{\pi}{2}\right]$$

——— (11)

$$\phi = \theta - \frac{\pi}{2}$$

$$\therefore \tan\phi = -\cot\theta$$

$$= -\frac{\frac{1}{LC} - p^2}{\frac{pR}{L}}$$

$$= \frac{p^2 - \frac{1}{LC}}{pR/L}$$

$$\tan\phi = \frac{pL - 1/pC}{R} \qquad \text{——— (a)}$$

$$\text{or,} \quad \phi = \tan^{-1}\left(\frac{pL - 1/pC}{R}\right) \qquad \text{——— (b)}$$

(12)

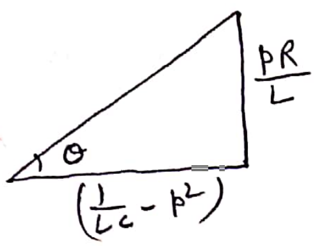
$\tan\theta = \dfrac{pR/L}{\left(\frac{1}{LC} - p^2\right)}$

$\therefore$ ~~tan~~ $=$

$\phi = \theta - \frac{\pi}{2}$

$\therefore \tan\phi = -\tan\left(\frac{\pi}{2} - \theta\right)$

$= -\cot\theta$



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eqn. (12) represents that current I lags in phase ϕ from E.M.F. E.

The quantity $\sqrt{R^2+(pL-\frac{1}{pC})^2}$ is called impedance of circuit and is represented by Z. Its unit is ohm.

Hence, $$I = \frac{E}{Z} = \frac{E_0 \sin(pt-\phi)}{Z}$$

$$I = I_0 \sin(pt-\phi) \qquad (13)$$

Where, $I_0 = \frac{E_0}{Z}$ is peak value of current.

The quantity $(pL - \frac{1}{pC})$ is called the reactance of the circuit and is denoted by X. Therefore, $X = pL - \frac{1}{pC}$

Impedance, $$Z = \sqrt{R^2+X^2}$$

$$Z = \sqrt{(\text{Resistance})^2 + (\text{Reactance})^2} \qquad (14)$$

current amplitude, $I_0 = \frac{E_0}{Z}$ is smaller for large values of Z.

The expression for $Z = \sqrt{R^2+(pL-\frac{1}{pC})^2}$ depends upon the frequency (p) of applied alternating E.M.F.

Now, we will discuss three cases -

Case I - $p << \omega_0$, the term $-\frac{1}{pC}$ in impedance is dominant and is very large. and

$$Z = \sqrt{R^2+(pL-\frac{1}{pC})^2} \approx \frac{1}{pC}$$

$\therefore$ current amplitude is (when $p \to 0$)

$$\left.\begin{aligned} I_0 &= \frac{E_0}{1/pC} \\ \text{and} \quad \phi &= -\frac{\pi}{2} \end{aligned}\right\} \qquad (15)$$

Thus for low values of p, current amplitude has a small value. Gradually increase in driven frequency (p) gives rise in current amplitude.

Case ii – When $p = \omega_0 = \frac{1}{\sqrt{LC}}$,

the reactance, $X = pL - \frac{1}{pC}$

$$= \frac{L}{\sqrt{LC}} - \frac{\sqrt{LC}}{C}$$

$$X = \sqrt{\frac{L}{C}} - \sqrt{\frac{L}{C}} = 0 \quad \text{—— (16)}$$

So, impedance Z is minimum and is equal to R, as a result the current amplitude $I_0 = \frac{E_0}{Z} = \frac{E_0}{R}$ becomes maximum. This is the case of electrical resonance. The resonant frequency of the circuit is given by.

$$p_r = \frac{p}{2\pi} = \frac{\omega_0}{2\pi}$$

$$\left.\begin{aligned} p_r &= \frac{1}{2\pi\sqrt{LC}} \\ \text{in such case, } \phi &= 0 \\ \text{and} \quad I &= \frac{E_0}{R}\sin pt \end{aligned}\right\} \quad \text{—— (17)}$$

Thus at resonance the current and the applied E.M.F. are in same phase.

The variation of I_0 with ω_0 is shown in fig. 1. The variation of phase difference ϕ between current and E.M.F. is shown in fig. 2.

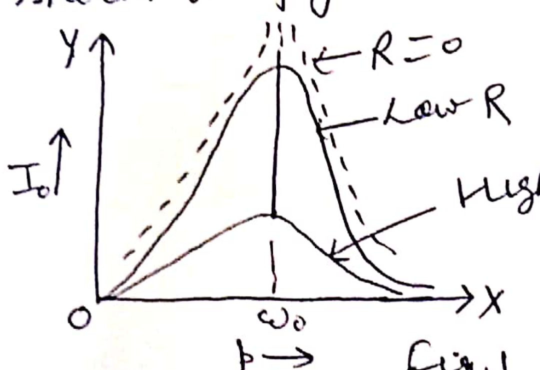

Fig. 1

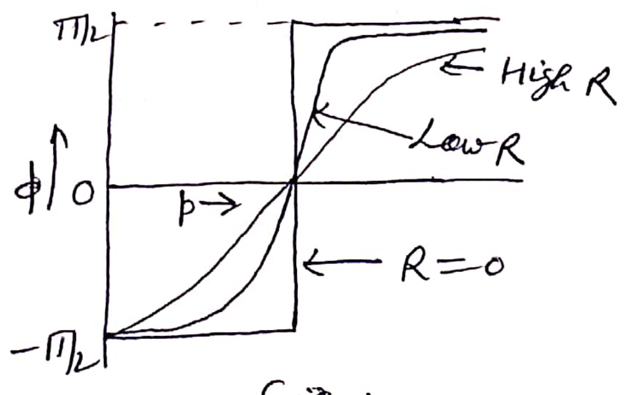

Fig 2

Case iii – When $p > \omega_0$, the term $L\omega$ in the expression for impedance $Z = \sqrt{R^2 + (pL - \frac{1}{pC})^2}$ becomes dominant and very large, and then

$$\left.\begin{aligned} I_0 &= \frac{E_0}{pL} \\ \text{and } \phi &= \frac{\pi}{2} \end{aligned}\right\} \quad \text{———} \quad (18)$$

The current lags in phase for $p < \omega_0$ and exceeds in phase for $p > \omega_0$.

Ques. – Define Power factor.

Ans - The average power of an A.C. circuit is given by

$$P_{av} = \frac{E_0 I_0}{2} \cos\phi$$

$$= \frac{E_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} \cos\phi$$

$$P_{av} = E_{rms} I_{rms} \cos\phi$$

This $\cos\phi$ is called power factor of the circuit and it will be given by –

$$\cos\phi = \frac{R}{\sqrt{R^2 + (pL - \frac{1}{pC})^2}}$$

$$= \frac{R}{Z}$$

at, resonance $\phi = 0$, so the average power delivered to the circuit is maximum.