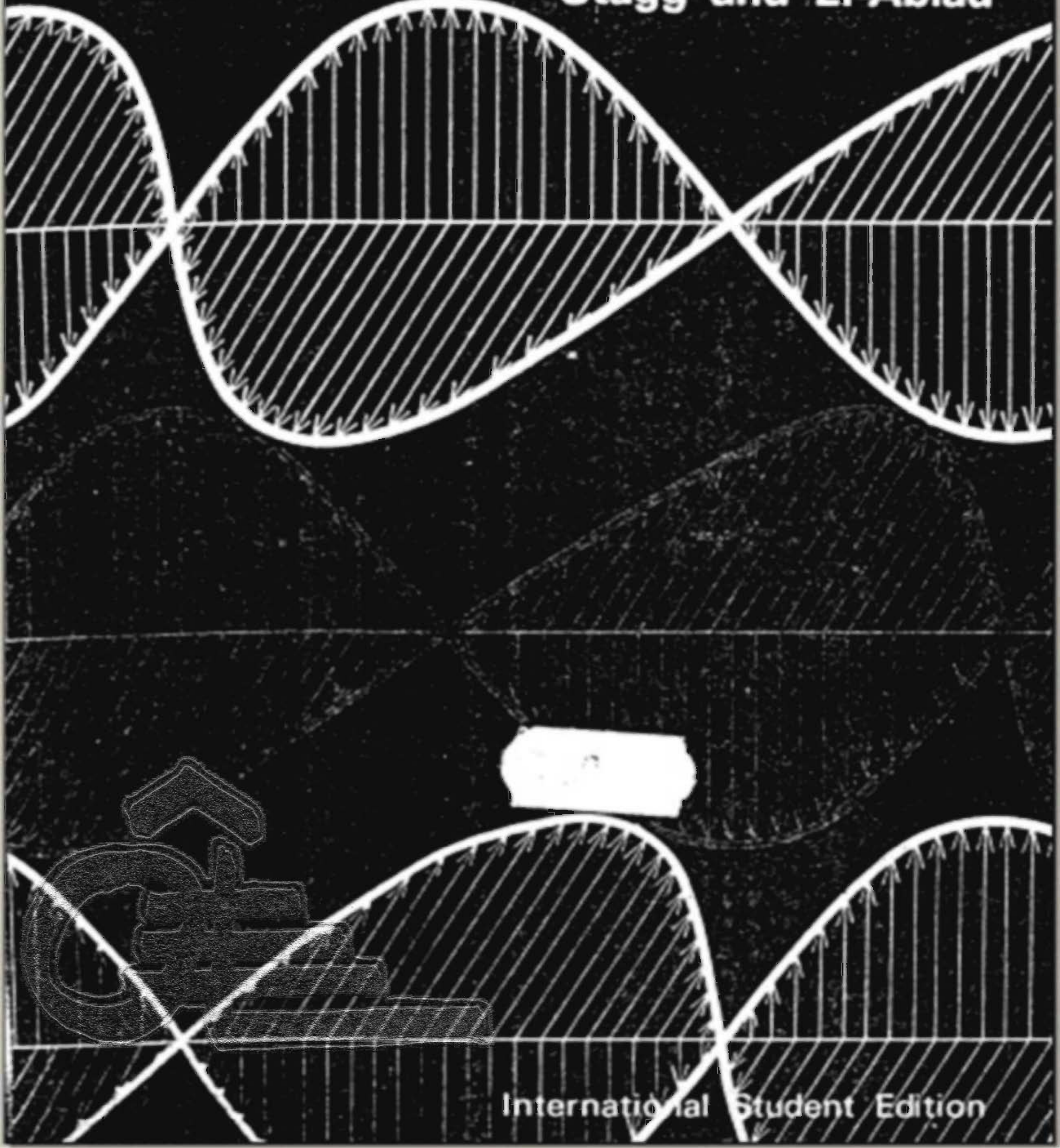
Computer Methods in Power System Analysis

Stagg and El-Abiad



International Student Edition

Computer Methods in Power System Analysis

JKT. 260394

92-0273.EL

McGraw-Hill Series in Electronic Systems

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Computer Methods in
Power System Analysis

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Library of Congress Catalog Card Number 67-12963

Preface

This book presents techniques that have been applied successfully in solving power system problems with a digital computer. It can thus serve as a text for advanced power system courses to inform prospective power engineers of methods currently employed in the electric utility industry. Because of the increasing use of the computer as an indispensable tool in power system engineering, this book will also serve as a basic reference for power system engineers responsible for the development of computer applications.

The material contained in the text has been developed from notes for special two-week courses offered since 1964 at Purdue University, The University of Wisconsin and the University of Santa Clara. These courses were attended by representatives of universities, electric utilities, and equipment manufacturers.

Solution techniques are presented for the three problems encountered most frequently in power system analysis, namely, short circuit, load flow, and power system stability. In addition to an engineering description of these problems, the mathematical techniques that are required for a computer solution are described. Thus, relevant material is included from matrix algebra and numerical analysis. It is assumed, however, that the reader has a general understanding of elementary power system analysis.

Chapter 1 presents, as a brief introduction, the impact of computers on power system engineering, the orientation of engineering problems to computers, and the advantages of digital computation. Chapter 2 covers the basic principles of matrix algebra and provides sufficient background in matrix theory for the remainder of the book. For readers familiar with matrix techniques, this chapter serves as a review and establishes the notation used throughout the text. Incidence and network matrices are introduced in Chapter 3, which presents the techniques for describing the geometric structure of a network and outlines the transformations required to derive network matrices. The formation of these matrices is the first step in the analysis of power system problems. Chapter 4 presents algorithms which can be used in an alternative method for the formation of certain network matrices. These algorithms have proved to be effective for use in computer calculation. The methods described in Chapters 3 and 4 are developed for single-phase representation of power systems. Chapter 5 extends these methods for three-phase representation. The application of network matrices to short circuit calculations is presented in Chapter 6. Several methods are included and a typical computer program is described to illustrate a practical application of the techniques.

Chapter 7 contains a brief introduction to the solution of linear and nonlinear simultaneous algebraic equations. This material is presented in a manner that affords direct application to the solution of the load flow problem. The formulation and solution of the load flow problem is presented in Chapter 8. This chapter also describes the procedures for handling voltage-controlled buses, transformers, and tie line control. The different methods are compared from several points of view and a description is given of an actual program used for load flow calculations. In a manner similar to that in Chapter 7, Chapter 9 introduces methods for the numerical solution of the differential equations that are required for transient stability studies. Chapter 10 covers the formulation and solution techniques employed in transient studies and presents procedures for the detailed representation of synchronous and induction machines, exciter and governor systems, and the distance relays. An actual transient stability computer program is described.

The first efforts in the development of this material were made in the early 1950s at the American Electric Power Service Corporation as a result of the interest in the application of computers to the planning and operation of electric power systems. In 1959, the authors had an opportunity to work together as members of the staff of the American Electric Power Service Corporation and continued to work together on a part-time basis for several years. This made possible the further development of basic computer methods established in previous years.

This research work was endorsed enthusiastically by the management of the American Electric Power Company. The authors wish to express their appreciation for this support.

It is a pleasure also to acknowledge the contributions of those who have helped in the preparation of this book. The authors would like particularly to thank Jorge F. Dopazo, who studied the text in detail and made many suggestions; Marjorie Watson, for her contribution related to the mathematical techniques and for editing the manuscript; and G. Robert Bailey, Dennis W. Johnston, Kasi Nagappan, Janice F. Hohenstein, and other members of the Engineering Analysis and Computer Division. The authors would like also to thank Professors Arun G. Phadke and Daniel K. Reitan of The University of Wisconsin for their helpful comments in reviewing the text. Last, but certainly not least, sincere thanks to Constance Aquila for her excellent work in the typing and general preparation of the manuscript.

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chapter 1
Introduction

1.1 Historical note

The great technical advances in the design and production of commercial and scientific general-purpose digital computers since the early 1950s have placed a powerful tool at the disposal of the engineering profession. This advancement has made economically feasible the utilization of digital computers for routine calculations encountered in everyday engineering work. In addition, it has provided the capability for performing more advanced engineering and scientific computations that were previously impossible because of their complex or time-consuming nature. All these trends have increased immensely the interest in digital computers and have necessitated a better understanding of the engineering and mathematical bases for problem solving.

The planning, design, and operation of power systems require continuous and comprehensive analysis to evaluate current system performance and to ascertain the effectiveness of alternative plans for system expansion. These studies play an important role in providing a high standard of power system reliability and ensuring the maximum utilization of capital investment.

The computational task of determining power flows and voltage levels resulting from a single operating condition for even a small network is all but insurmountable if performed by manual methods. The need for computational aids in power system engineering led in 1929 to the design of a special-purpose analog computer called an ac network analyzer. This device made possible the study of a greater variety of system operating conditions for both present and future system designs. It provided the ability to determine power flows and system voltages during normal and emergency conditions and to study the transient behavior of the system resulting from fault conditions and switching operations. By the middle 1950s 50 network analyzers were in operation in the United States and Canada and were indispensable tools to planning, relaying, and operating engineers.

The earliest application of digital computers to power system problems dates back to the late 1940s. However, most of the early applications were limited in scope because of the small capacity of the punched card calculators generally in use at that time. The availability of large-scale digital computers in the middle 1950s provided equipment of sufficient capacity and speed to meet the requirements of major power system problems. In 1957 the American Electric Power Service Corporation completed a large-scale load flow program for the IBM 704 which calculated the voltages and power flows for a specified power system network.

The initial application of the load flow program to transmission planning studies proved so successful that all subsequent studies employed the digital computer instead of the network analyzer. The success of this program led to the development of programs for short circuit and transient stability calculations. Today the computer is an indispensable tool in all phases of power system planning, design, and operation.

1.2 *Impact of computers*

The development of computer technology has provided the following advantages to power system engineering:

1. More efficient and economic means of performing routine engineering calculations required in the planning, design, and operation of a power system
2. A better utilization of engineering talent by relieving the engineer from tedious hand calculations and permitting him to spend more time on technical work
3. The ability to perform more effective engineering studies by applying calculating procedures to obtain a number of alternate solutions for a particular problem to provide a broad base for engineering decisions
4. The capability of performing studies which heretofore were not possible because of the volume of calculations involved

Two major factors which have contributed to the realization of these benefits are the declining cost of computing equipment and the development of efficient computational techniques. Now that a substantial reduction in computing cost has been effected, principal effort must be directed toward the orientation of engineering problems to computer solutions.

1.3 Orientation of engineering problems to computers

The process of applying a computer to the solution of engineering problems involves a number of distinct steps. These steps are:

1. Problem definition Initially, the problem must be defined precisely and the objectives determined. This may be the most difficult step in the entire process. Consideration must be given to the pertinent data available for input, the scope of the problem and its limitations, the desired results, and their relative importance in making an engineering decision. This phase requires the judgement of experienced and capable engineers.

2. Mathematical formulation After the problem has been defined, it is necessary to develop a mathematical model to represent the physical system. This requires specifying the characteristics of individual system components as well as the relations which govern the interconnection of the elements. Different mathematical models may be used to represent the same system and, for many problems, complementary (dual) formulations may be obtained. One formulation may result in a different number of equations than another as, for example, in the case of network problems which can be solved using either loop equations or node equations. The mathematical formulation of the problem, therefore, includes the design of a number of models and the selection of the best model to describe the physical system.

3. Selection of a solution technique The formulation of most engineering problems involves mathematical expressions, such as sets of nonlinear equations, differential equations, and trigonometric functions, which cannot be evaluated directly by a digital computer. A computer is capable of performing only the four basic arithmetic operations of addition, subtraction, multiplication, and division. A solution for any problem, therefore, must be obtained by numerical techniques which employ the four basic arithmetic operations. It is important in this phase to select a method which is practical for machine computation and, in particular, will produce the desired results in a reasonable amount of computer time. Since numerical approaches involve a number of assumptions, careful consideration must be given to the degree of accuracy required.

4. Program design The sequence of logical steps by which a particular problem is to be solved, the allocation of memory, the access of data, and the assignment of input and output units are important aspects of computer program design. The objectives are primarily to develop a procedure which eliminates unnecessary repetitive calculations and remains within the capability of the computer. The program design is usually prepared in the form of a diagram called a flow chart.

5. *Programming* A digital computer has a series of instructions consisting of operation codes and addresses which it is able to interpret and execute. In addition to the arithmetic and input/output instructions, logical instructions are available which are used to direct the sequence of calculations. The translation of the precise detailed steps to be performed in the solution of a problem into an organized list of computer instructions is the process of programming. A program can be developed by using computer instructions in actual or symbolic form, or it can be written in a generalized programming language, such as FORTRAN.

6. *Program verification* There are many opportunities to introduce errors in the development of a complete computer program. Therefore, a systematic series of checks must be performed to ensure the correctness of problem formulation, method of solution, and operation of the program.

7. *Application* Engineering programs, in general, can be classified into two groups. The first consists of special-purpose programs, which are developed in a relatively short period for the solution of simple engineering problems. Such a problem is usually well defined, and often the program completely serves its purpose after the first series of calculations has been completed. Some small programs are used on a continuing basis but are restricted in their use because of their special-purpose nature.

The second group consists of general-purpose programs that are designed for the analysis of large engineering problems. These programs are applied extensively in the regular studies of one or more engineering departments. Their use may have an effect on the approach to an engineering problem and the organization of a study. Thus, it is important that consideration be given to the manner in which a program is to be employed in an engineering activity. Some aspects which must be considered are means of collecting and preparing data, processing time, and presentation of results. Programs of this type are becoming an integral part of power system engineering.

The relative importance of each of these steps varies from problem to problem. Moreover, all steps are closely related and play an important role in the decisions that must be made. Of primary importance is the interrelation of the mathematical formulation of a problem and the selection of a solution technique. Frequently, it is difficult to evaluate the influence of these two steps on each other without developing a complete program and performing actual calculations to compare the alternatives.

The material covered in this book pertains to the first three steps, with particular emphasis on the interrelations of steps 2 and 3. Simplified flow charts are used to illustrate the methods presented.

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chapter 2

Matrix algebra

2.1 Introduction

In recent years, the use of matrix algebra for the formulation and solution of complex engineering problems has become increasingly important with the advent of digital computers to perform the required calculations. The application of matrix notation provides a concise and simplified means of expressing many problems. The use of matrix operations presents a logical and ordered process which is readily adaptable for a computer solution of a large system of simultaneous equations.

2.2 Basic concepts and definitions

Matrix notation

Matrix notation is a shorthand means of writing systems of simultaneous equations in a concise form. A matrix is defined as a rectangular array of numbers, called *elements*, arranged in a systematic manner with m rows and n columns. These elements can be real or complex numbers. A double-subscript notation a_{ij} is used to designate a matrix element. The first subscript i designates the row in which the element lies, and the second subscript j designates the column.

In the following system of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= y_3 \end{aligned} \tag{2.2.1}$$

x_1, x_2, and x_3 are unknown variables; a_{11}, a_{12}, a_{13}, . . . , a_{33} are the coefficients of these variables; y_1, y_2, and y_3 are known parameters. The coefficients form an array

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{2.2.2}$$

which is the *coefficient matrix* of the system of equations (2.2.1).

Similarly, the variables and parameters can be written in matrix form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{2.2.3}$$

The matrix (2.2.2) is designated by a capital letter A and the matrices (2.2.3) by X and Y, respectively. In matrix notation the equations (2.2.1) are written

$$AX = Y$$

A matrix with m rows and n columns is said to be of dimension m by n, or $m \times n$. A matrix with a single row and more than one column ($m = 1$ and $n > 1$) is called a *row matrix* or *row vector*. A matrix with a single column and more than one row is called a *column matrix* or *column vector*.

Types of matrices

Some matrices with special characteristics are significant in matrix operations. These are:

Square matrix When the number of rows equals the number of columns, that is, $m = n$, the matrix is called a *square matrix* and its order is equal to the number of rows (or columns). The elements in a square matrix a_{ij} for which $i = j$ are called ***diagonal elements***. Those for which $i \neq j$ are called ***off-diagonal elements***. For elements a_{ij} to the right of the diagonal i is less than j, and for those to the left of the diagonal i is greater than j.

Upper triangular matrix If the elements a_{ij} of a square matrix are zero for $i > j$, then the matrix is an *upper triangular matrix*. For example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Lower triangular matrix If the elements a_{ij} of a square matrix are zero for $i < j$, then the matrix is a *lower triangular matrix.* For example:

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Diagonal matrix If all off-diagonal elements of a square matrix are zero ($a_{ij} = 0$ for all $i \neq j$), then the matrix is a *diagonal matrix.* For example:

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Unit or identity matrix If all diagonal elements of a square matrix equal one and all other elements are zero ($a_{ij} = 1$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$), the matrix is the *unit* or *identity matrix*, designated by the letter U. For example:

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null matrix If all elements of a matrix are zero, it is a *null matrix.*

Transpose of a matrix If the rows and columns of an $m \times n$ matrix are interchanged, the resultant $n \times m$ matrix is the *transpose* and is designated by A^t. For the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

the transpose is

$$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

Symmetric matrix If the corresponding off-diagonal elements of a square matrix are equal ($a_{ij} = a_{ji}$), the matrix is a *symmetric matrix.* For example:

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 2 & 6 \\ 3 & 6 & 4 \end{bmatrix}$$

The transpose of a symmetric matrix is identical to the matrix itself, that is, $A^t = A$.

Skew-symmetric matrix If $A = -A^t$ for a square matrix, A is a *skew-symmetric matrix.* The corresponding off-diagonal elements are equal but of opposite sign ($a_{ij} = -a_{ji}$) and the diagonal elements are zero. For example:

$$A = \begin{bmatrix} 0 & -5 & 3 \\ 5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

Orthogonal matrix If $A^tA = U = AA^t$ for a square matrix with real elements, then A is an *orthogonal matrix.*

Conjugate of a matrix If all the elements of a matrix are replaced by their conjugates (replace the element $a + jb$ by $a - jb$), the resultant matrix is the *conjugate* and is designated by A^*. For a matrix

$$A = \begin{bmatrix} j3 & 5 \\ 4 + j2 & 1 + j1 \end{bmatrix}$$

the conjugate is

$$A^* = \begin{bmatrix} -j3 & 5 \\ 4 - j2 & 1 - j1 \end{bmatrix}$$

If all the elements of A are real, then $A = A^*$. If all elements are pure imaginary, then $A = -A^*$.

Hermitian matrix If $A = (A^*)^t$ for a square complex matrix, A is a *Hermitian matrix* in which all diagonal elements are real. For example:

$$A = \begin{bmatrix} 4 & 2 - j3 \\ 2 + j3 & 5 \end{bmatrix}$$

Skew-Hermitian matrix If $A = -(A^*)^t$ for a square complex matrix, A is a *skew-Hermitian matrix* in which all diagonal elements are either zero or pure imaginary. For example:

$$A = \begin{bmatrix} 0 & 2 - j3 \\ -2 - j3 & 0 \end{bmatrix}$$

Unitary matrix If $(A^*)^tA = U = A(A^*)^t$ for a square complex matrix, A is a *unitary matrix.* A unitary matrix with real elements is an orthogonal matrix.

Table 2.1 summarizes some types of special matrices.

Table 2.1 Types of special matrices

Condition	*Type of matrix*
$A = -A$	Null
$A = A^t$	Symmetric
$A = -A^t$	Skew-symmetric
$A = A^*$	Real
$A = -A^*$	Pure imaginary
$A = (A^*)^t$	Hermitian
$A = -(A^*)^t$	Skew-Hermitian
$A^tA = U$	Orthogonal
$(A^*)^tA = U$	Unitary

2.3 Determinants

Definition and properties of determinants

The solution of two simultaneous equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= k_1 \\ a_{21}x_1 + a_{22}x_2 &= k_2 \end{aligned} \tag{2.3.1}$$

can be obtained by eliminating the variables one at a time. Solving for x_2 in terms of x_1 from the second equation and substituting this expression for x_2 in the first equation, the following is obtained:

$$\begin{aligned} a_{11}x_1 + a_{12}\left(\frac{k_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1\right) &= k_1 \\ a_{11}a_{22}x_1 + a_{12}k_2 - a_{12}a_{21}x_1 &= a_{22}k_1 \\ (a_{11}a_{22} - a_{12}a_{21})x_1 &= a_{22}k_1 - a_{12}k_2 \\ x_1 &= \frac{a_{22}k_1 - a_{12}k_2}{a_{11}a_{22} - a_{12}a_{21}} \end{aligned}$$

Then, substituting x_1 in either of the equations (2.3.1), x_2 is obtained:

$$x_2 = \frac{a_{11}k_2 - a_{21}k_1}{a_{11}a_{22} - a_{12}a_{21}}$$

The expression $(a_{11}a_{22} - a_{12}a_{21})$ is the value of the determinant of the coefficient matrix A, where $|A|$ denotes the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

The solution of the equations (2.3.1) by means of determinants is

$$x_1 = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{22}k_1 - a_{12}k_2}{a_{11}a_{22} - a_{12}a_{21}}$$

and

$$x_2 = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11}k_2 - a_{21}k_1}{a_{11}a_{22} - a_{12}a_{21}}$$

A determinant is defined only for a square matrix and has a single value. A method for evaluating the determinant of an $n \times n$ matrix is given in Chap. 7.

Determinants have the following properties:

1. The value of a determinant is zero if
    a. All elements of a row or column are zero
    b. The corresponding elements of two rows (or columns) are equal
    c. A row (or column) is a linear combination of one or more rows (or columns)
2. If two rows (or columns) of a determinant are interchanged, the value of the determinant is changed in sign only
3. The value of a determinant is not changed if
    a. All corresponding rows and columns are interchanged, i.e.,

       $$|A| = |A^t|$$

    b. k times the elements of any row (or column) are added to the corresponding elements of another row (or column)
4. If all elements of a row (or column) are multiplied by a factor k, the value of the determinant is multiplied by k
5. The determinant of the product of matrices is equal to the product of the determinants of the matrices, i.e.,

   $$|A\,B\,C| = |A|\,|B|\,|C|$$

6. The determinant of the sum (or difference) of matrices is *not* equal to the sum (or difference) of the individual determinants, i.e.,

   $$|A + B - C| \neq |A| + |B| - |C|$$

The application of these properties can reduce the work in evaluating determinants.

Minors and cofactors

The determinant obtained by striking out the ith row and jth column is called the *minor* of the element a_{ij}. Thus, for

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

the minor of a_{21} is

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

The order of this minor is one less than that of the original determinant. By striking out any two rows and columns a minor of order two less than the original determinant is obtained, etc.

The *cofactor* of an element is

$$(-1)^{i+j}(\text{minor of } a_{ij})$$

where the order of the minor of a_{ij} is $n - 1$. The cofactor of a_{21}, designated by A_{21}, is

$$A_{21} = (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

The following relationships between a determinant and cofactors exist:

1. The sum of the products of the elements in any row (or column) and their cofactors is equal to the determinant:

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \tag{2.3.2}$$

2. The sum of the products of the elements in any row (or column) and the cofactors of the corresponding elements in another row (or column) is equal to zero:

$$a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} = 0 \tag{2.3.3}$$

Adjoint

If each element of a square matrix is replaced by its cofactor and then the matrix is transposed, the resulting matrix is an *adjoint* which is designated by A^+

$$A^+ = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

2.4 Matrix operations

Equality of matrices

If A and B are matrices with the same dimension and each element a_{ij} of A is equal to the corresponding element b_{ij} of B, the matrices are equal, i.e.,

$$A = B$$

Addition and subtraction of matrices

Matrices of the same dimension are conformable for addition and subtraction. The sum or difference of two $m \times n$ matrices, A and B, is a matrix C of the same dimension, i.e.,

$$A \pm B = C$$

where each element of C is

$$c_{ij} = a_{ij} \pm b_{ij}$$

For n conformable matrices the sum or difference is

$$A \pm B \pm C \pm D \pm \cdots \pm N = R$$

where the elements of the resultant matrix R are

$$r_{ij} = a_{ij} \pm b_{ij} \pm c_{ij} \pm d_{ij} \pm \cdots \pm n_{ij}$$

The commutative and associative laws apply to addition of matrices as follows:

$A + B = B + A$ commutative law

i.e., the sum of the matrices is independent of the order of the addition.

$A + B + C = A + (B + C) = (A + B) + C$ associative law

i.e., the sum of the matrices is independent of the order in which the matrices are associated for addition.

Multiplication of a matrix by a scalar

When a matrix is multiplied by a scalar, the elements of the resultant matrix are equal to the product of the original elements and the scalar. For example:

$$kA = B$$

where $b_{ij} = ka_{ij}$ for all i and j.

The multiplication of a matrix by a scalar obeys the commutative law and the distributive law as follows:

$kA = Ak$ commutative law
$k(A + B) = kA + kB = (A + B)k$ distributive law

Multiplication of matrices

Multiplication of two matrices

$$AB = C$$

is defined only if the number of columns of the first matrix A equals the number of rows of B. Thus, for the product of matrix A of dimension $m \times q$ and matrix B of dimension $q \times n$, the matrix C is of dimension $m \times n$. Any element c_{ij} of C is the sum of the products of the corresponding elements of the ith row of A and the jth column of B, that is,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{iq}b_{qj}$$

or

$$c_{ij} = \sum_{k=1}^{q} a_{ik}b_{kj} \qquad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$$

For example:

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

In the product AB, A premultiplies B or B postmultiplies A. The product BA is not defined since the number of columns of B is not equal to the number of rows of A. When the products AB and BA are defined for a square matrix, it can be shown that, in general,

$$AB \neq BA$$

Therefore, the commutative law does not hold for matrix multiplication. If the matrices A, B, and C satisfy the dimension requirements for multiplication and addition, the following properties hold:

$A(B + C) = AB + AC$ distributive law
$A(BC) = (AB)C = ABC$ associative law

However,

$AB = 0$ does not necessarily imply that $A = 0$ or $B = 0$
$CA = CB$ does not necessarily imply that $A = B$

If $C = AB$, then the transpose of C is equal to the product of the transposed matrices in reverse order, i.e.,

$$C^t = B^t A^t$$

This is the *reversal rule.*

Inverse of a matrix

Division does not exist in matrix algebra except in the case of the division of a matrix by a scalar. This operation is performed by dividing each element of a matrix by the scalar. However, for a given set of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= y_3 \end{aligned} \tag{2.4.1}$$

or, in matrix form,

$$AX = Y \tag{2.4.2}$$

it is desirable to express x_1, x_2, and x_3 as functions of y_1, y_2, and y_3, that is,

$$X = BY$$

If there is a unique solution for the equations (2.4.1), then matrix B exists and is the *inverse* of A.

If the determinant of A is not zero, the equations can be solved for the x_i's as follows:

$$\begin{aligned} x_1 &= \frac{A_{11}}{|A|} y_1 + \frac{A_{21}}{|A|} y_2 + \frac{A_{31}}{|A|} y_3 \\ x_2 &= \frac{A_{12}}{|A|} y_1 + \frac{A_{22}}{|A|} y_2 + \frac{A_{32}}{|A|} y_3 \\ x_3 &= \frac{A_{13}}{|A|} y_1 + \frac{A_{23}}{|A|} y_2 + \frac{A_{33}}{|A|} y_3 \end{aligned}$$

where A_{11}, A_{12}, . . . , A_{33} are the cofactors of a_{11}, a_{12}, . . . , a_{33} and $|A|$ is the determinant of A. Thus

$$B = \begin{bmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \frac{A_{31}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \frac{A_{32}}{|A|} \\ \frac{A_{13}}{|A|} & \frac{A_{23}}{|A|} & \frac{A_{33}}{|A|} \end{bmatrix} = \frac{A^+}{|A|}$$

where A^+ is the adjoint of A. It should be noted that the elements of the adjoint A^+ are the cofactors of the elements of A, but are placed in transposed position. The matrix B is the inverse of A and is written A^{-1}.

Multiplying A by its inverse,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \dfrac{A_{11}}{|A|} & \dfrac{A_{21}}{|A|} & \dfrac{A_{31}}{|A|} \\ \dfrac{A_{12}}{|A|} & \dfrac{A_{22}}{|A|} & \dfrac{A_{32}}{|A|} \\ \dfrac{A_{13}}{|A|} & \dfrac{A_{23}}{|A|} & \dfrac{A_{33}}{|A|} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

results in the unit matrix. This follows from the relationships (2.3.2) and (2.3.3). A diagonal term of U, such as u_{11}, equals 1 since

$$a_{11}\frac{A_{11}}{|A|} + a_{12}\frac{A_{12}}{|A|} + a_{13}\frac{A_{13}}{|A|} = \frac{|A|}{|A|} = 1$$

and an off-diagonal term, such as u_{12}, equals zero since

$$a_{11}\frac{A_{21}}{|A|} + a_{12}\frac{A_{22}}{|A|} + a_{13}\frac{A_{23}}{|A|} = \frac{0}{|A|} = 0$$

Thus

$$AA^{-1} = A^{-1}A = U$$

To solve for X from the matrix equation (2.4.2) both sides of the equation are premultiplied by A^{-1}.

$$\begin{aligned} AX &= Y \\ A^{-1}AX &= A^{-1}Y \\ UX &= A^{-1}Y \\ X &= A^{-1}Y \end{aligned}$$

The order of the matrices in the product must be maintained.

If the determinant of a matrix is zero, the inverse does not exist. Such a matrix is called a *singular matrix*. If the determinant of a matrix is not zero, the matrix is a *nonsingular matrix* and has a unique inverse.

The inverse of the product of matrices can be obtained by the reversal rule, i.e.,

$$(AB)^{-1} = B^{-1}A^{-1}$$

The transpose and inverse operations on a matrix can be interchanged, i.e.,

$$(A^t)^{-1} = (A^{-1})^t$$

Partitioning of matrices

A large matrix can be subdivided into several submatrices of smaller dimensions:

$$\left[\begin{array}{c} \\ A \\ \\ \end{array}\right] = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]$$

If the diagonal submatrices A_1 and A_4 are square, the subdivision is called *principal partitioning.*

Partitioning can be used to show the specific structure of A and to simplify matrix computation. Each submatrix is considered as an element in the partitioned matrix. Addition or subtraction is performed as follows:

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \pm \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1 \pm B_1 & A_2 \pm B_2 \\ \hline A_3 \pm B_3 & A_4 \pm B_4 \end{array}\right]$$

where the dimensions of corresponding submatrices must be conformable.

Multiplication is performed as follows:

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right]$$

where

$$C_1 = A_1B_1 + A_2B_3$$
$$C_2 = A_1B_2 + A_2B_4$$
$$C_3 = A_3B_1 + A_4B_3$$
$$C_4 = A_3B_2 + A_4B_4$$

The rule for partitioning two matrices whose product is to be found is: the n columns of the premultiplier are grouped into k and $n - k$ columns from left to right, and the n rows of the postmultiplier are grouped into k and $n - k$ rows from top to bottom in order that the submatrices are conformable for multiplication.

The transpose of a partitioned matrix is shown below.

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]$$

$$A^t = \left[\begin{array}{c|c} A_1^t & A_3^t \\ \hline A_2^t & A_4^t \end{array}\right]$$

The inverse of a partitioned matrix is obtained as follows:

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]$$

$$A^{-1} = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]$$

where

$$\begin{aligned} B_1 &= (A_1 - A_2A_4^{-1}A_3)^{-1} \\ B_2 &= -B_1A_2A_4^{-1} \\ B_3 &= -A_4^{-1}A_3B_1 \\ B_4 &= A_4^{-1} - A_4^{-1}A_3B_2 \end{aligned} \tag{2.4.3.}$$

and A_1 and A_4 must be square matrices.

2.5 Linear dependence and rank of a matrix

Linear dependence

The columns of an $m \times n$ matrix A can be written as n column vectors.

$\{c_1\}\{c_2\} \cdots \{c_n\}$

Also, the rows of matrix A can be written as m row vectors.

$\{r_1\}\{r_2\} \cdots \{r_m\}$

The column vectors are *linearly independent* if the equation

$$p_1\{c_1\} + p_2\{c_2\} + \cdots + p_n\{c_n\} = 0 \tag{2.5.1}$$

is satisfied only for all $p_k = 0$ $(k = 1, 2, \ldots, n)$. Similarly, the row vectors are linearly independent if only zero values for the scalars q_r $(r = 1, 2, \ldots, m)$ satisfy the equation

$$q_1\{r_1\} + q_2\{r_2\} + \cdots + q_m\{r_m\} = 0 \tag{2.5.2}$$

It is not possible to express one or more linearly independent column vectors (or row vectors) as a linear combination of others.

If some $p_k \neq 0$ satisfies (2.5.1), the column vectors are *linearly dependent.* If some $q_r \neq 0$ satisfies (2.5.2), the row vectors are linearly dependent. That is, it is possible to express one or more column vectors (or row vectors) as a linear combination of others. If the column vectors (or row vectors) of a matrix A are linearly dependent, then the determinant of A is zero.

Rank of a matrix

The *rank* of an $m \times n$ matrix A is equal to the maximum number of linearly independent columns of A or the maximum number of linearly independent rows of A. The former is called the *column rank* and the latter the *row rank.* The column rank is equal to the row rank. The rank of a matrix is equal to the order of the largest nonvanishing determinant in A. For example, consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 8 & 10 \end{bmatrix}$$

The rows are linearly dependent since the equation

$$q_1\{1\ 2\ 4\} + q_2\{2\ 4\ 8\} + q_3\{3\ 8\ 10\} = 0$$

is satisfied for

$$q_1 = 2$$
$$q_2 = -1$$
$$q_3 = 0$$

Similarly, the columns are linearly dependent since the equation

$$p_1 \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} + p_2 \begin{Bmatrix} 2 \\ 4 \\ 8 \end{Bmatrix} + p_3 \begin{Bmatrix} 4 \\ 8 \\ 10 \end{Bmatrix} = 0$$

is satisfied for

$$p_1 = 6$$
$$p_2 = -1$$
$$p_3 = -1$$

However, no two columns are linearly dependent and, therefore, the rank of the matrix is 2.

2.6 Linear equations

A linear system of m equations in n unknowns is written

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= y_m \end{aligned} \qquad (2.6.1)$$

where a_{ij} = known coefficients or parameters of the system
x_j = unknown variables of the system
y_i = known constants of the system

The system of equations (2.6.1) in matrix form is

$$AX = Y$$

The *augmented matrix* of A, designated by $\hat{A}$, is formed by adjoining the column vector Y as the $(n + 1)$st column to A.

$$\hat{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_m \end{bmatrix}$$

If $y_1, y_2, \ldots, y_m$ are all zero in (2.6.1), the linear equations are *homogeneous* and

$$AX = 0$$

If one or more y_i are nonzero, the linear equations are *nonhomogeneous.*

The necessary and sufficient condition for a system of linear equations to have a solution is that the rank of the coefficient matrix A be equal to the rank of the augmented matrix $\hat{A}$. A unique solution exists when A is a square matrix and the rank of A is equal to the number of columns (variables). The unique solution is nontrivial for nonhomogeneous equations and trivial (i.e., zero) for homogeneous equations. If the rank of A is less than the number of equations, some of the equations are redundant and do not place any further constraint on the variables. If the rank of A is less than the number of variables of the system, there are an infinite number of nontrivial solutions.

Problems

2.1 Given:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 3 & 2 & 4 \\ -7 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 5 \\ -3 & 4 & 4 \\ 7 & -2 & 2 \end{bmatrix}$$

Determine:

a. $C = A + B$

b. What type of matrix C is

c. $D = A - B$

d. What type of matrix D is

2.2 Given:

$$A = \begin{bmatrix} -3 & 1 & 4 \\ -2 & -5 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Determine:

a. $C = AB$

b. $D = BA$

c. $E = A^tB^t$

d. What the relationship of matrix E to matrix D is

2.3 Given:

$$A = \begin{bmatrix} 11 & 8 & 5 \\ 8 & 1 & 5 \\ 5 & 5 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$$

Determine:

a. $C = B^tAB$

b. What type of matrix A is

c. What type of matrix C is

d. $D = C^{-1}$

e. $E = CC^{-1}$

f. What type of matrix E is

2.4 Given:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Determine:

a. $B = A^{-1}$

b. X from $AX = Y$

2.5 Given:

$$A = \begin{bmatrix} 0 & 5 & 7 & 1 \\ -5 & 0 & -1 & 7 \\ -7 & 1 & 0 & -5 \\ -1 & -7 & 5 & 0 \end{bmatrix}$$

Determine:

a. $B = 0.1A + 0.5U$

b. What type of matrix A is

c. That B is orthogonal

2.6 Given:

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}$$

Determine:

a. $B = (A^*)^t A$

b. What type of matrix A is

2.7 Given:

$$A = \begin{bmatrix} 2 & -j2 & 1 + j3 \\ j2 & 1 & 2 - j \\ 1 - j3 & 2 + j & 3 \end{bmatrix}$$

Determine:

a. $B = (A^*)^t$

b. What type of matrix A is

2.8 Given:

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix} \qquad jB = \begin{bmatrix} j2 & j \\ j4 & 0 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

Determine:

a. X from $(A + jB)X^* = Y$

b. What type of matrix A is

c. What type of matrix B is

2.9 Given the partitioned matrix:

$$A = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ -1 & -2 & 1 \\ \hline -6 & 4 & 2 \end{array}\right]$$

Determine $B = A^{-1}$ using the formulas (2.4.3) for the inverse of a partitioned matrix.

2.10 Given:

$$\left[\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 3 & 4 \\ 2 & 6 & 0 & 0 & 1 \\ \hline 0 & 0 & 14 & 5 & 1 \\ 3 & 0 & 5 & 10 & 2 \\ 4 & 1 & 1 & 2 & 12 \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \\ 1 \\ 1 \end{array}\right]$$

Determine A_2.

2.11 Given the partitioned matrices:

$$A = \left[\begin{array}{cc|ccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 5 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

$$B = \left[\begin{array}{cc|ccc|c} 1 & 3 & 2 & 5 & 6 & 1 \\ 4 & 7 & 1 & 3 & 2 & 4 \\ \hline 3 & 4 & 2 & 6 & 5 & 1 \\ 4 & 6 & 3 & 1 & 2 & 5 \\ 2 & 6 & 7 & 3 & 8 & 1 \\ \hline 7 & 2 & 3 & 1 & 4 & 5 \end{array}\right]$$

Determine $C = AB$.

2.12 Given:

$$C = \left[\begin{array}{c|c} A & N \\ \hline N & B \end{array}\right]$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ 6 & -4 \end{bmatrix}$$

and N is a null matrix.
Show that the inverse of C is

$$C^{-1} = \left[\begin{array}{c|c} A^{-1} & N \\ \hline N & B^{-1} \end{array}\right]$$

2.13 Given:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 5 & 6 & 3 \\ 3 & 5 & 4 \end{bmatrix}$$

Show that A is a singular matrix and determine its rank.

2.14 Given:

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] = \left[\begin{array}{ccc|c} 6 & 0 & 3 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 7 & 5 & 3 \\ \hline 6 & 4 & 2 & 2 \end{array}\right]$$

Determine $B = A_1 - A_2A_4^{-1}A_3$.

2.15 Given:

$$A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 5 & -8 \\ 1 & -8 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \qquad \text{and}$$

$$C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Determine $D = A - C'BC$.

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chapter 3
Incidence and network matrices

3.1 *Introduction*

The formulation of a suitable mathematical model is the first step in the analysis of an electrical network. The model must describe the characteristics of individual network components as well as the relations that govern the interconnection of these elements. A network matrix equation provides a convenient mathematical model for a digital computer solution.

The elements of a network matrix depend on the selection of the independent variables, which can be either currents or voltages. Correspondingly, the elements of the network matrix will be impedances or admittances.

The electrical characteristics of the individual network components can be presented conveniently in the form of a primitive network matrix. This matrix, while adequately describing the characteristics of each component, does not provide any information pertaining to the network connections. It is necessary, therefore, to transform the primitive network matrix into a network matrix that describes the performance of the interconnected network.

The form of the network matrix used in the performance equation depends on the frame of reference, namely, bus or loop. In the bus frame of reference the variables are the nodal voltages and nodal currents. In the loop frame of reference the variables are loop voltages and loop currents.

The formation of the appropriate network matrix is an integral part of a digital computer program for the solution of power system problems.

3.2 Graphs

In order to describe the geometrical structure of a network it is sufficient to replace the network components by single line segments irrespective of the characteristics of the components. These line segments are called *elements* and their terminals are called *nodes.* A node and an element are *incident* if the node is a terminal of the element. Nodes can be incident to one or more elements.

A *graph* shows the geometrical interconnection of the elements of a network. A *subgraph* is any subset of elements of the graph. A *path* is a subgraph of connected elements with no more than two elements connected to any one node. A graph is *connected* if and only if there is a path between every pair of nodes. If each element of the connected graph is assigned a direction it is then *oriented.* A representation of a power system and the corresponding oriented graph are shown in Fig. 3.1.

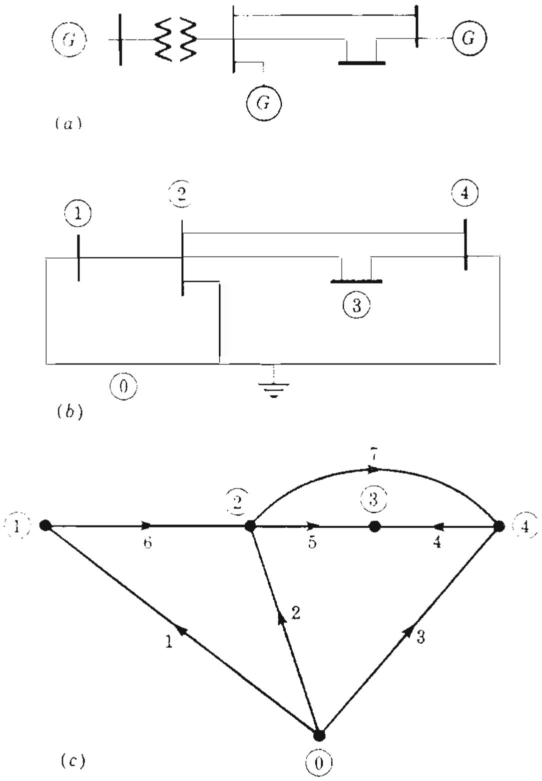


Fig. 3.1 Power system representations. (a) Single line diagram; (b) positive sequence network diagram; (c) oriented connected graph.

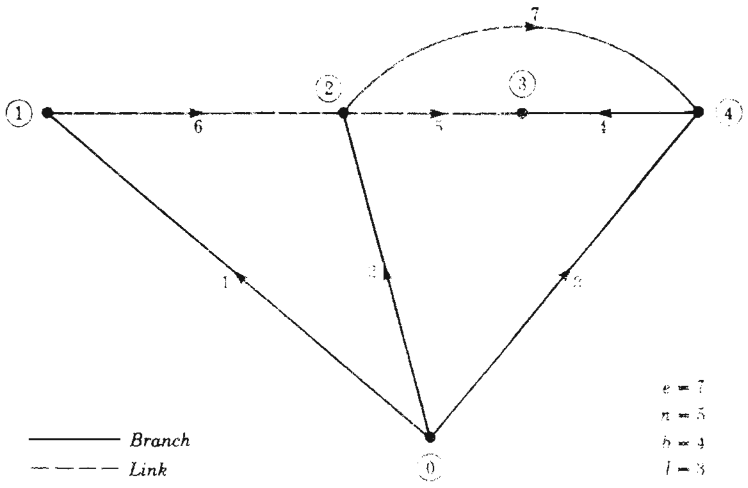


Fig. 3.2 Tree and cotree of the oriented connected graph.

A connected subgraph containing all nodes of a graph but no closed path is called a *tree*. The elements of a tree are called *branches* and form a subset of the elements of the connected graph. The number of branches b required to form a tree is

$$b = n - 1 \tag{3.2.1}$$

where n is the number of nodes in the graph.

Those elements of the connected graph that are not included in the tree are called *links* and form a subgraph, not necessarily connected, called the *cotree*. The cotree is the complement of the tree. The number of links l of a connected graph with e elements is

$$l = e - b$$

From equation (3.2.1) it follows that

$$l = e - n + 1 \tag{3.2.2}$$

A tree and the corresponding cotree of the graph given in Fig. 3.1c are shown in Fig. 3.2.

If a link is added to the tree, the resulting graph contains one closed path, called a *loop*. The addition of each subsequent link forms one or more additional loops. Loops which contain only one link are independent and are called *basic loops*. Consequently, the number of basic

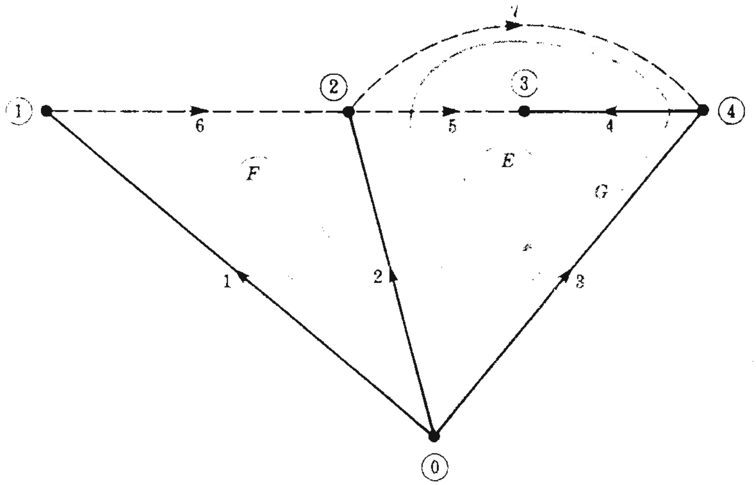


Fig. 3.3 Basic loops of the oriented connected graph.

loops is equal to the number of links given by equation (3.2.2). Orientation of a basic loop is chosen to be the same as that of its link. The basic loops of the graph given in Fig. 3.2 are shown in Fig. 3.3.

A *cut-set* is a set of elements that, if removed, divides a connected graph into two connected subgraphs. A unique independent group of

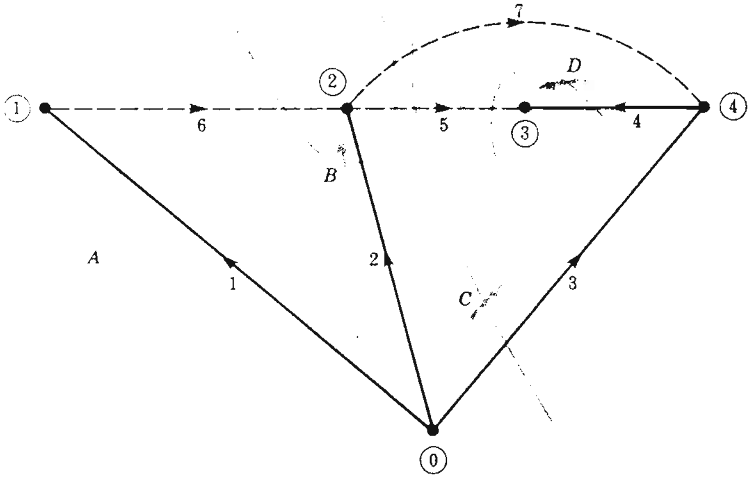


Fig. 3.4 Basic cut-sets of the oriented connected graph.

cut-sets may be chosen if each cut-set contains only one branch. Independent cut-sets are called *basic cut-sets*. The number of basic cut-sets is equal to the number of branches. Orientation of a basic cut-set is chosen to be the same as that of its branch. The basic cut-sets of the graph given in Fig. 3.2 are shown in Fig. 3.4.

3.3 Incidence matrices

Element-node incidence matrix $\hat{A}$

The incidence of elements to nodes in a connected graph is shown by the element-node incidence matrix. The elements of the matrix are as follows:

$a_{ij} = 1$ if the ith element is incident to and oriented away from the jth node

$a_{ij} = -1$ if the ith element is incident to and oriented toward the jth node

$a_{ij} = 0$ if the ith element is not incident to the jth node

The dimension of the matrix is $e \times n$, where e is the number of elements and n is the number of nodes in the graph. The element-node incidence matrix for the graph shown in Fig. 3.2 is

$$\hat{A} = \begin{array}{c|c|c|c|c|c|}
e \backslash n & \textcircled{0} & \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\
\hline
1 & 1 & -1 & & & \\
\hline
2 & 1 & & -1 & & \\
\hline
3 & 1 & & & & -1 \\
\hline
4 & & & & -1 & 1 \\
\hline
5 & & & 1 & -1 & \\
\hline
6 & & 1 & -1 & & \\
\hline
7 & & & 1 & & -1 \\
\hline
\end{array}$$

Since

$$\sum_{j=0}^{4} a_{ij} = 0 \qquad i = 1, 2, \ldots, e$$

the columns of $\hat{A}$ are linearly dependent. Hence, the rank of $\hat{A} < n$.

Bus incidence matrix A

Any node of a connected graph can be selected as the reference node. Then, the variables of the other nodes, referred to as buses, can be measured with respect to the assigned reference. The matrix obtained from $\hat{A}$ by deleting the column corresponding to the reference node is the element-bus incidence matrix A, which will be called the bus incidence matrix. The dimension of this matrix is $e \times (n - 1)$ and the rank is $n - 1 = b$, where b is the number of branches in the graph. Selecting node 0 as reference for the graph shown in Fig. 3.2,

$$A = \begin{array}{c|c|c|c|c|}
{}_{e}\diagdown^{\text{bus}} & \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\
\hline
1 & -1 & & & \\
\hline
2 & & -1 & & \\
\hline
3 & & & & -1 \\
\hline
4 & & & -1 & 1 \\
\hline
5 & & 1 & -1 & \\
\hline
6 & 1 & -1 & & \\
\hline
7 & & 1 & & -1 \\
\hline
\end{array}$$

This matrix is rectangular and therefore singular.

If the rows of A are arranged according to a particular tree, the matrix can be partitioned into submatrices A_b of dimension $b \times (n - 1)$ and A_l of dimension $l \times (n - 1)$, where the rows of A_b correspond to branches and the rows of A_l to links. The partitioned matrix for the graph shown in Fig. 3.2 is

$$A = \begin{array}{c|cccc}
e \backslash \text{bus} & ① & ② & ③ & ④ \\
\hline
1 & -1 & & & \\
2 & & -1 & & \\
3 & & & & -1 \\
4 & & & -1 & 1 \\
5 & & 1 & -1 & \\
6 & 1 & -1 & & \\
7 & & 1 & & -1
\end{array}
=
\begin{array}{c|c}
e \backslash \text{bus} & \text{Buses} \\
\hline
\text{Branches} & A_b \\
\hline
\text{Links} & A_l
\end{array}$$

A_b is a nonsingular square matrix with rank $(n - 1)$.

Branch-path incidence matrix K

The incidence of branches to paths in a tree is shown by the branch-path incidence matrix, where a path is oriented from a bus to the reference node. The elements of this matrix are:

k_{ij} = 1 if the ith branch is in the path from the jth bus to reference and is oriented in the same direction

k_{ij} = -1 if the ith branch is in the path from the jth bus to reference but is oriented in the opposite direction

k_{ij} = 0 if the ith branch is not in the path from the jth bus to reference

With node 0 as reference the branch-path incidence matrix associated with the tree shown in Fig. 3.2 is

$$K = \begin{array}{c|cccc}
b \backslash \text{path} & ① & ② & ③ & ④ \\
\hline
1 & -1 & & & \\
2 & & -1 & & \\
3 & & & -1 & -1 \\
4 & & & -1 &
\end{array}$$

This is a nonsingular square matrix with rank $(n - 1)$.

The branch-path incidence matrix and the submatrix A_b relate the branches to paths and branches to buses, respectively. Since there is a one-to-one correspondence between paths and buses,

$$A_b K^t = U \tag{3.3.1}$$

Therefore,

$$K^t = A_b^{-1} \tag{3.3.2}$$

Basic cut-set incidence matrix B

The incidence of elements to basic cut-sets of a connected graph is shown by the basic cut-set incidence matrix B. The elements of this matrix are:

b_{ij} = 1 if the ith element is incident to and oriented in the same direction as the jth basic cut-set
b_{ij} = -1 if the ith element is incident to and oriented in the opposite direction as the jth basic cut-set
b_{ij} = 0 if the ith element is not incident to the jth basic cut-set

The basic cut-set incidence matrix, of dimension $e \times b$, for the graph shown in Fig. 3.4 is

$$B = \begin{array}{c|c|c|c|c} e \backslash b & \multicolumn{4}{c}{\text{Basic cut-sets}} \\ & A & B & C & D \\ \hline 1 & 1 & & & \\ \hline 2 & & 1 & & \\ \hline 3 & & & 1 & \\ \hline 4 & & & & 1 \\ \hline 5 & & -1 & 1 & 1 \\ \hline 6 & -1 & 1 & & \\ \hline 7 & & -1 & 1 & \end{array}$$

The matrix B can be partitioned into submatrices U_b and B_l where the rows of U_b correspond to branches and the rows of B_l to links. The partitioned matrix is

$$B = \begin{array}{c|cccc} e \backslash b & \multicolumn{4}{c}{\text{Basic cut-sets}} \\ & A & B & C & D \\ \hline 1 & 1 & & & \\ 2 & & 1 & & \\ 3 & & & 1 & \\ 4 & & & & 1 \\ \hline 5 & & -1 & 1 & 1 \\ 6 & -1 & 1 & & \\ 7 & & -1 & 1 & \end{array} = \begin{array}{c|c} e \backslash b & \text{Basic cut-sets} \\ \hline \text{Branches} & U_b \\ \hline \text{Links} & B_l \end{array}$$

The identity matrix U_b shows the one-to-one correspondence of the branches and basic cut-sets.

The submatrix B_l can be obtained from the bus incidence matrix A. The incidence of links to buses is shown by the submatrix A_l and the incidence of branches to buses is shown by the submatrix A_b. Since there is a one-to-one correspondence of the branches and basic cut-sets, $B_l A_b$ shows the incidence of links to buses. that is,

$$B_l A_b = A_l$$

Therefore,

$$B_l = A_l A_b^{-1}$$

In addition, as shown in equation (3.3.2),

$$A_b^{-1} = K^t$$

Therefore,

$$B_l = A_l K^t \tag{3.3.3}$$

Augmented cut-set incidence matrix $\hat{B}$

Fictitious cut-sets, called *tie cut-sets*, can be introduced in order that the number of cut-sets equals the number of elements. Each tie cut-set contains only one link of the connected graph. The tie cut-sets for the graph given in Fig. 3.4 are shown in Fig. 3.5. An augmented cut-set incidence matrix is formed by adjoining to the basic cut-set incidence matrix additional columns corresponding to these tie cut-sets. A tie

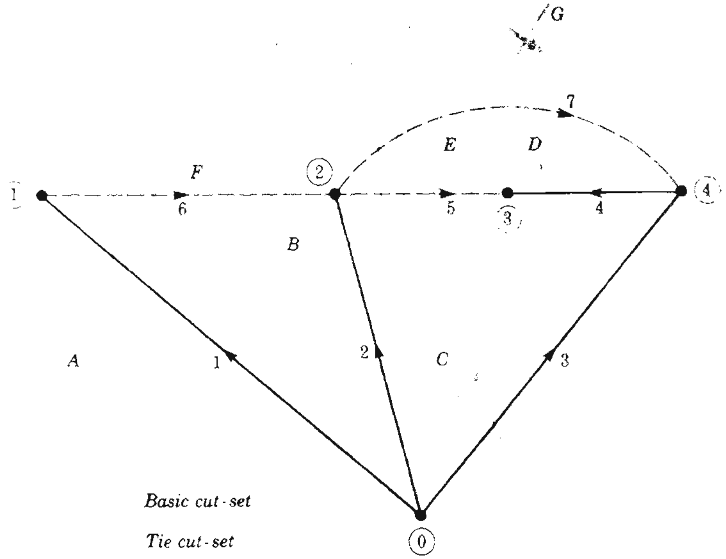


Fig. 3.5 Basic and tie cut-sets of the oriented connected graph.

cut-set is oriented in the same direction as the associated link. The augmented cut-set incidence matrix for the graph shown in Fig. 3.5 is

$$\hat{B} = \begin{array}{c|ccccccc} & \multicolumn{4}{c}{\text{Basic cut-sets}} & \multicolumn{3}{c}{\text{Tie cut-sets}} \\ e \backslash e & A & B & C & D & E & F & G \\ \hline 1 & 1 & & & & & & \\ 2 & & 1 & & & & & \\ 3 & & & 1 & & & & \\ 4 & & & & 1 & & & \\ 5 & & -1 & 1 & 1 & 1 & & \\ 6 & -1 & 1 & & & & 1 & \\ 7 & & -1 & 1 & & & & 1 \end{array}$$

This is a square matrix of dimension $e \times e$ and is nonsingular.

The matrix $\hat{B}$ can be partitioned as follows:

$$\hat{B} = \begin{array}{c|cccc|ccc}
 & \multicolumn{4}{c|}{\text{Basic cut-sets}} & \multicolumn{3}{c}{\text{Tie cut-sets}} \\
e \backslash e & A & B & C & D & E & F & G \\
\hline
1 & 1 & & & & & & \\
2 & & 1 & & & & & \\
3 & & & 1 & & & & \\
4 & & & & 1 & & & \\
\hline
5 & & -1 & 1 & 1 & 1 & & \\
6 & -1 & 1 & & & & 1 & \\
7 & & -1 & 1 & & & & 1
\end{array}$$

$$= \begin{array}{c|c|c}
e \backslash e & \text{Basic cut-sets} & \text{Tie cut-sets} \\
\hline
\text{Branches} & U_b & 0 \\
\hline
\text{Links} & B_l & U_l
\end{array}$$

Basic loop incidence matrix C

The incidence of elements to basic loops of a connected graph is shown by the basic loop incidence matrix C. The elements of this matrix are:

$c_{ij} = 1$ if the ith element is incident to and oriented in the same direction as the jth basic loop

$c_{ij} = -1$ if the ith element is incident to and oriented in the opposite direction as the jth basic loop

$c_{ij} = 0$ if the ith element is not incident to the jth basic loop

The basic loop incidence matrix, of dimension $e \times l$, for the graph shown in Fig. 3.3 is

$$C = \begin{array}{c|ccc} e \diagdown l & & \text{Basic loops} & \\ & E & F & G \\ \hline 1 & & 1 & \\ 2 & 1 & -1 & 1 \\ 3 & -1 & & -1 \\ 4 & -1 & & \\ 5 & 1 & & \\ 6 & & 1 & \\ 7 & & & 1 \end{array}$$

The matrix C can be partitioned into submatrices C_b and U_l where the rows of C_b correspond to branches and the rows of U_l to links. The partitioned matrix is

$$C = \begin{array}{c|ccc} e \diagdown l & & \text{Basic loops} & \\ & E & F & G \\ \hline 1 & & 1 & \\ 2 & 1 & -1 & 1 \\ 3 & -1 & & -1 \\ 4 & -1 & & \\ \hline 5 & 1 & & \\ 6 & & 1 & \\ 7 & & & 1 \end{array} = \begin{array}{c|c} e \diagdown l & \text{Basic loops} \\ \hline \text{Branches} & C_b \\ \hline \text{Links} & U_l \end{array}$$

The identity matrix U_l shows the one-to-one correspondence of links to basic loops.

Augmented loop incidence matrix $\hat{C}$

The number of basic loops in a connected graph is equal to the number of links. In order to have a total number of loops equal to the number of

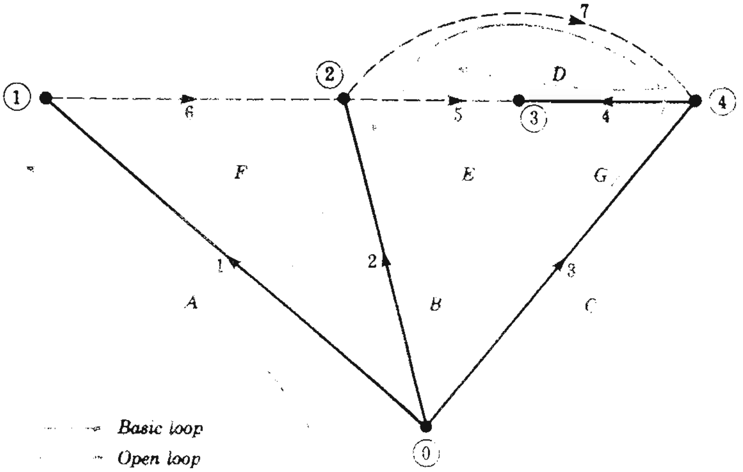


Fig. 3.6 Basic and open loops of the oriented connected graph.

elements, let $(e - b)$ loops, corresponding to the b branches, be designated as *open loops*. An open loop, then, is defined as a path between adjacent nodes connected by a branch. The open loops for the graph given in Fig. 3.3 are shown in Fig. 3.6. The orientation of an open loop is the same as that for the associated branch.

The augmented loop incidence matrix is formed by adjoining to the basic loop incidence matrix the columns showing the incidence of elements to open loops. This matrix, for the graph shown in Fig. 3.6, is

$$\hat{C} = \begin{array}{c|cccc|ccc} e \backslash e & \multicolumn{4}{c|}{\text{Open loops}} & \multicolumn{3}{c}{\text{Basic loops}} \\ & A & B & C & D & E & F & G \\ \hline 1 & 1 & & & & & 1 & \\ 2 & & 1 & & & 1 & -1 & 1 \\ 3 & & & 1 & & -1 & & -1 \\ 4 & & & & 1 & -1 & & \\ 5 & & & & & 1 & & \\ 6 & & & & & & 1 & \\ 7 & & & & & & & 1 \end{array}$$

This is a square matrix, of dimension $e \times e$, and is nonsingular.

The matrix $\hat{C}$ can be partitioned as follows:

$$\hat{C} = \begin{array}{c|cccc|ccc}
 & \multicolumn{4}{c|}{\text{Open loops}} & \multicolumn{3}{c}{\text{Basic loops}} \\
e \backslash e & A & B & C & D & E & F & G \\
\hline
1 & 1 & & & & & 1 & \\
2 & & 1 & & & 1 & -1 & 1 \\
3 & & & 1 & & -1 & & -1 \\
4 & & & & 1 & -1 & & \\
\hline
5 & & & & & 1 & & \\
6 & & & & & & 1 & \\
7 & & & & & & & 1
\end{array}$$

$$= \begin{array}{c|c|c}
e \backslash e & \text{Open loops} & \text{Basic loops} \\
\hline
\text{Branches} & U_b & C_b \\
\hline
\text{Links} & 0 & U_l
\end{array}$$

3.4 Primitive network

Network components represented both in impedance form and in admittance form are shown in Fig. 3.7. The performance of the components can be expressed using either form. The variables and parameters are:

v_{pq} is the voltage across the element p–q

e_{pq} is the source voltage in series with element p–q

i_{pq} is the current through element p–q

j_{pq} is the source current in parallel with element p–q
z_{pq} is the self-impedance of element p–q
y_{pq} is the self-admittance of element p–q

Each element has two variables, v_{pq} and i_{pq}. In steady state these variables and the parameters of the elements z_{pq} and y_{pq} are real numbers for direct current circuits and complex numbers for alternating current circuits.

The performance equation of an element in impedance form is

$$v_{pq} + e_{pq} = z_{pq}i_{pq} \tag{3.4.1}$$

or in admittance form is

$$i_{pq} + j_{pq} = y_{pq}v_{pq} \tag{3.4.2}$$

The parallel source current in admittance form is related to the series source voltage in impedance form by

$$j_{pq} = -y_{pq}e_{pq}$$

A set of unconnected elements is defined as a primitive network. The performance equations of a primitive network can be derived from (3.4.1) or (3.4.2) by expressing the variables as vectors and the parame-

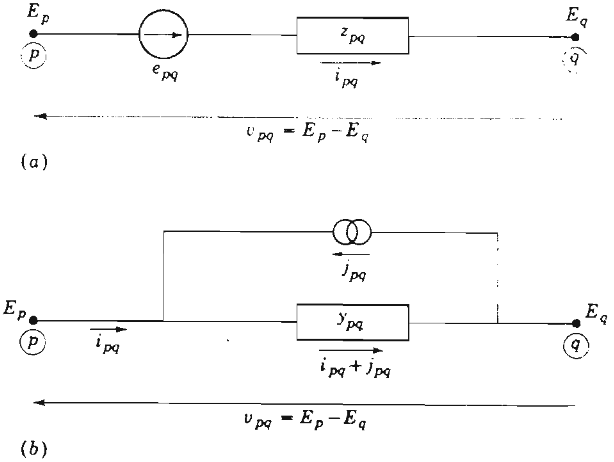


Fig. 3.7 Representations of a network component. (a) Impedance form; (b) admittance form.

ters as matrices. The performance equation in impedance form is

$$\bar{v} + \bar{e} = [z]\bar{i}$$

or in admittance form is

$$\bar{i} + \bar{j} = [y]\bar{v}$$

A diagonal element of the matrix $[z]$ or $[y]$ of the primitive network is the self-impedance $z_{pq,pq}$ or self-admittance $y_{pq,pq}$. An off-diagonal element is the mutual impedance $z_{pq,rs}$ or the mutual admittance $y_{pq,rs}$ between the elements p–q and r–s. The primitive admittance matrix $[y]$ can be obtained by inverting the primitive impedance matrix $[z]$. The matrices $[z]$ and $[y]$ are diagonal matrices if there is no mutual coupling between elements. In this case the self-impedances are equal to the reciprocals of the corresponding self-admittances.

3.5 Formation of network matrices by singular transformations

Network performance equations

A network is made up of an interconnected set of elements. In the bus frame of reference, the performance of an interconnected network is described by $n - 1$ independent nodal equations, where n is the number of nodes. In matrix notation, the performance equation in impedance form is

$$\bar{E}_{BUS} = Z_{BUS}\bar{I}_{BUS}$$

or in admittance form is

$$\bar{I}_{BUS} = Y_{BUS}\bar{E}_{BUS}$$

where $\bar{E}_{BUS}$ = vector of bus voltages measured with respect to the reference bus

$\bar{I}_{BUS}$ = vector of impressed bus currents

Z_{BUS} = bus impedance matrix whose elements are open circuit driving point and transfer impedances

Y_{BUS} = bus admittance matrix whose elements are short circuit driving point and transfer admittances

In the branch frame of reference the performance of the interconnected network is described by b independent branch equations where b is the number of branches. In matrix notation, the performance equation in impedance form is

$$\bar{E}_{BR} = Z_{BR}\bar{I}_{BR}$$

or in admittance form is

$$\bar{I}_{BR} = Y_{BR}\bar{E}_{BR}$$

where $\bar{E}_{BR}$ = vector of voltages across the branches
$\bar{I}_{BR}$ = vector of currents through the branches
Z_{BR} = branch impedance matrix whose elements are open circuit driving point and transfer impedances of the branches of the network
Y_{BR} = branch admittance matrix whose elements are short circuit driving point and transfer admittances of the branches of the network

In the loop frame of reference, the performance of an interconnected network is described by l independent loop equations where l is the number of links or basic loops. The performance equation in impedance form is

$$\bar{E}_{LOOP} = Z_{LOOP}\bar{I}_{LOOP}$$

or in admittance form is

$$\bar{I}_{LOOP} = Y_{LOOP}\bar{E}_{LOOP}$$

where $\bar{E}_{LOOP}$ = vector of basic loop voltages
$\bar{I}_{LOOP}$ = vector of basic loop currents
Z_{LOOP} = loop impedance matrix
Y_{LOOP} = loop admittance matrix

Bus admittance and bus impedance matrices

The bus admittance matrix Y_{BUS} can be obtained by using the bus incidence matrix A to relate the variables and parameters of the primitive network to bus quantities of the interconnected network. The performance equation of the primitive network

$$\bar{\imath} + \bar{\jmath} = [y]\bar{v}$$

is premultiplied by A^t, the transpose of the bus incidence matrix, to obtain

$$A^t\bar{\imath} + A^t\bar{\jmath} = A^t[y]\bar{v} \tag{3.5.1}$$

Since the matrix A shows the incidence of elements to buses, $A^t\bar{\imath}$ is a vector in which each element is the algebraic sum of the currents through the network elements terminating at a bus. In accordance with Kirchhoff's current law, the algebraic sum of the currents at a bus is zero. Then

$$A^t\bar{\imath} = 0 \tag{3.5.2}$$

Similarly, $A^t\bar{\jmath}$ gives the algebraic sum of the source currents at each bus and equals the vector of impressed bus currents. Therefore

$$\bar{I}_{BUS} = A^t\bar{\jmath} \tag{3.5.3}$$

Substituting from equations (3.5.2) and (3.5.3) into (3.5.1) yields

$$\bar{I}_{BUS} = A^t[y]\bar{v} \tag{3.5.4}$$

Power into the network is $(\bar{I}^*_{BUS})^t\bar{E}_{BUS}$ and the sum of the powers in the primitive network is $(\bar{\jmath}^*)^t\bar{v}$. The power in the primitive and interconnected networks must be equal, that is, the transformation of variables must be power-invariant. Hence

$$(\bar{I}^*_{BUS})^t\bar{E}_{BUS} = (\bar{\jmath}^*)^t\bar{v} \tag{3.5.5}$$

Taking the conjugate transpose of equation (3.5.3)

$$(\bar{I}^*_{BUS})^t = (\bar{\jmath}^*)^t A^*$$

Since A is a real matrix

$$A^* = A$$

and

$$(\bar{I}^*_{BUS})^t = (\bar{\jmath}^*)^t A \tag{3.5.6}$$

Substituting from equation (3.5.6) into (3.5.5)

$$(\bar{\jmath}^*)^t A\bar{E}_{BUS} = (\bar{\jmath}^*)^t\bar{v}$$

Since this equation is valid for all values of $\bar{\jmath}$, it follows that

$$A\bar{E}_{BUS} = \bar{v} \tag{3.5.7}$$

Substituting from equation (3.5.7) into (3.5.4),

$$\bar{I}_{BUS} = A^t[y]A\bar{E}_{BUS} \tag{3.5.8}$$

Since the performance equation of the network is

$$\bar{I}_{BUS} = Y_{BUS}\bar{E}_{BUS} \tag{3.5.9}$$

it follows from equations (3.5.8) and (3.5.9) that

$$Y_{BUS} = A^t[y]A$$

The bus incidence matrix A is singular and therefore $A^t[y]A$ is a singular transformation of $[y]$.

The bus impedance matrix can be obtained from

$$Z_{BUS} = Y^{-1}_{BUS} = (A^t[y]A)^{-1}$$

Branch admittance and branch impedance matrices

The branch admittance matrix Y_{BR} can be obtained by using the basic cut-set incidence matrix B to relate the variables and parameters of the primitive network to branch quantities of the interconnected network. The performance equation of the primitive network in admittance form is premultiplied by B^t to obtain

$$B^t\bar{\imath} + B^t\bar{\jmath} = B^t[y]\bar{v} \tag{3.5.10}$$

Since the matrix B shows the incidence of elements to basic cut-sets, $B^t\bar{\imath}$ is a vector in which each element is the algebraic sum of the currents through the elements incident to a basic cut-set.

The elements of a basic cut-set if removed divide the network into two connected subnetworks. Therefore, an element of the vector $B^t\bar{\imath}$ is the algebraic sum of the current entering a subnetwork and by Kirchhoff's current law is zero. Therefore

$$B^t\bar{\imath} = 0 \tag{3.5.11}$$

Similarly, $B^t\bar{\jmath}$ is a vector in which each element is the algebraic sum of the source currents of the elements incident to the basic cut-set and is the total source current in parallel with a branch. Therefore

$$\bar{I}_{BR} = B^t\bar{\jmath} \tag{3.5.12}$$

Substituting from equations (3.5.11) and (3.5.12) into (3.5.10) yields

$$\bar{I}_{BR} = B^t[y]\bar{v} \tag{3.5.13}$$

Power into the network is $(\bar{I}^*_{BR})^t(\bar{E}_{BR})$ and since power is invariant

$$(\bar{I}^*_{BR})^t\bar{E}_{BR} = (\bar{\jmath}^*)^t\bar{v}$$

Obtaining $(\bar{I}^*_{BR})^t$ from equation (3.5.12), then

$$(\bar{\jmath}^*)^t B^*\bar{E}_{BR} = (\bar{\jmath}^*)^t\bar{v}$$

Since B is a real matrix

$$B^* = B \qquad \text{and} \qquad (\bar{\jmath}^*)^t B\bar{E}_{BR} = (\bar{\jmath}^*)^t\bar{v}$$

Since this equation is valid for all values of $\bar{\jmath}$, it follows that

$$\bar{v} = B\bar{E}_{BR} \tag{3.5.14}$$

Substituting from equation (3.5.14) into (3.5.13) yields

$$\bar{I}_{BR} = B^t[y]B\bar{E}_{BR} \tag{3.5.15}$$

The relation between the branch currents and the branch voltages is

$$\bar{I}_{BR} = Y_{BR}\bar{E}_{BR} \tag{3.5.16}$$

It follows from equations (3.5.15) and (3.5.16) that

$$Y_{BR} = B^t[y]B$$

The basic cut-set matrix B is a singular matrix and therefore $B^t[y]B$ is a singular transformation of $[y]$.

The branch impedance matrix can be obtained from

$$Z_{BR} = Y_{BR}^{-1} = (B^t[y]B)^{-1}$$

Loop impedance and loop admittance matrices

The loop impedance matrix Z_{LOOP} can be obtained by using the basic loop incidence matrix C to relate the variables and parameters of the primitive network to loop quantities of the interconnected network. The performance equation of the primitive network

$$\bar{v} + \bar{e} = [z]\bar{\imath}$$

is premultiplied by C^t to obtain

$$C^t\bar{v} + C^t\bar{e} = C^t[z]\bar{\imath} \qquad (3.5.17)$$

Since the matrix C shows the incidence of elements to basic loops, $C^t\bar{v}$ gives the algebraic sum of the voltages around each basic loop. In accordance with Kirchhoff's voltage law, the algebraic sum of the voltages around a loop is zero. Hence

$$C^t\bar{v} = 0 \qquad (3.5.18)$$

Similarly $C^t\bar{e}$ gives the algebraic sum of the source voltages around each basic loop. Therefore

$$\bar{E}_{LOOP} = C^t\bar{e} \qquad (3.5.19)$$

Since power is invariant

$$(\bar{I}^*_{LOOP})^t\bar{E}_{LOOP} = (\bar{\imath}^*)^t\bar{e}$$

Substituting for $\bar{E}_{LOOP}$ from equation (3.5.19), then

$$(\bar{I}^*_{LOOP})^t C^t\bar{e} = (\bar{\imath}^*)^t\bar{e}$$

Since this equation is valid for all values of $\bar{e}$, it follows that

$$(\bar{\imath}^*)^t = (\bar{I}^*_{LOOP})^t C^t$$

Hence,

$$\bar{\imath} = C^*\bar{I}_{LOOP}$$

Since C is a real matrix,

$$C^* = C$$

and

$$\bar{i} = C\bar{I}_{LOOP} \tag{3.5.20}$$

Substituting from equations (3.5.18), (3.5.19), and (3.5.20) into (3.5.17) yields

$$\bar{E}_{LOOP} = C^t[z]C\bar{I}_{LOOP} \tag{3.5.21}$$

The performance equation of the network in the loop frame of reference is

$$\bar{E}_{LOOP} = Z_{LOOP}\bar{I}_{LOOP} \tag{3.5.22}$$

and it follows from equations (3.5.21) and (3.5.22) that

$$Z_{LOOP} = C^t[z]C$$

Since C is a singular matrix, $C^t[z]C$ is a singular transformation of $[z]$.

Table 3.1 Formation of network matrices by singular transformations

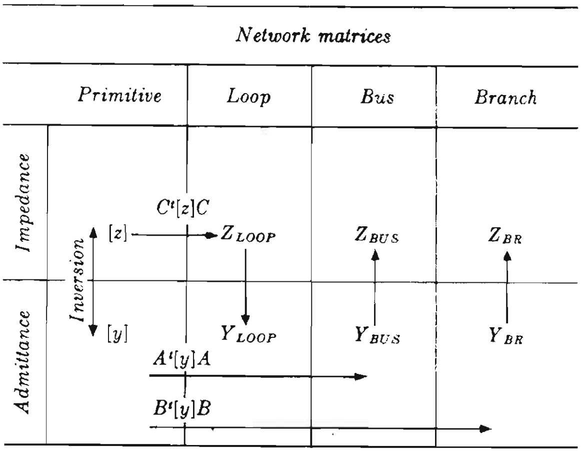


Table 3.2 Current and voltage relations between primitive and interconnected networks

	Frame of reference		
	Loop	Bus	Branch
Current	$\bar{i} = C\bar{I}_{LOOP}$	$\bar{I}_{BUS} = A^t\bar{j}$	$\bar{I}_{BR} = B^t\bar{j}$
Voltage	$\bar{E}_{LOOP} = C^t\bar{e}$	$\bar{v} = A\bar{E}_{BUS}$	$\bar{v} = B\bar{E}_{BR}$

The loop admittance matrix can be obtained from

$$Y_{LOOP} = Z_{LOOP}^{-1} = (C^t[z]C)^{-1}$$

The singular transformations for obtaining network matrices are summarized in Table 3.1. The current and voltage relations between the primitive and interconnected networks are summarized in Table 3.2.

3.6 Formation of network matrices by nonsingular transformations

Branch admittance and branch impedance matrices

The branch admittance matrix Y_{BR} can be obtained also by using the augmented cut-set incidence matrix $\hat{B}$ to relate the variables and parameters of the primitive network to those of an augmented interconnected network. The augmented network is obtained by connecting a fictitious branch in series with each link of the original network. In order to preserve the performance of the interconnected network the admittance of each fictitious branch is set to zero and its current source is set equal to the current through the associated link, as shown in Fig. 3.8*a*. The voltage across a fictitious branch is zero. Then a tie cut-set can be interpreted as a cut-set containing a link and a fictitious branch, as shown in Fig. 3.8*b*.

The performance equation of the augmented network in the branch frame of reference is

$$\hat{I}_{BR} = \hat{Y}_{BR}\hat{E}_{BR}$$

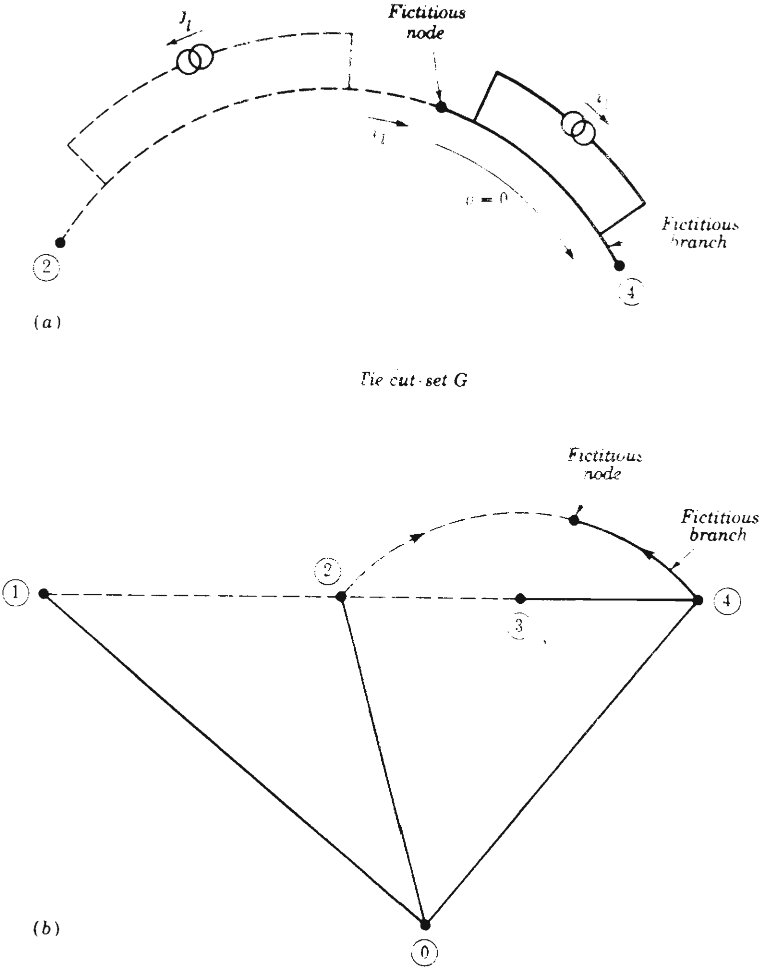


Fig. 3.8 Representation of an augmented network. (a) Fictitious branch in series with a link; (b) interpretation of a tie cut-set.

The matrix Y_{BR} will be obtained directly from the admittance matrix $\hat{Y}_{BR}$ of the augmented network.

The performance equation for the primitive network

$$\bar{\imath} + \bar{\jmath} = [y]\bar{v}$$

is premultiplied by $\hat{B}^t$ to obtain

$$\hat{B}^t\bar{\imath} + \hat{B}^t\bar{\jmath} = \hat{B}^t[y]\bar{v} \qquad (3.6.1)$$

Equation (3.6.1) can be written in the partitioned matrix form:

$$\left[\begin{array}{c|c} U_b & B_l^t \\ \hline 0 & U_l \end{array}\right]\left[\begin{array}{c} i_b \\ \hline i_l \end{array}\right] + \left[\begin{array}{c|c} U_b & B_l^t \\ \hline 0 & U_l \end{array}\right]\left[\begin{array}{c} j_b \\ \hline j_l \end{array}\right]$$

$$= \left[\begin{array}{c|c} U_b & B_l^t \\ \hline 0 & U_l \end{array}\right]\left[\begin{array}{c} y \end{array}\right]\left[\begin{array}{c} v \end{array}\right] \qquad (3.6.2)$$

where the primitive current vectors $\bar{\imath}$ and $\bar{\jmath}$ are partitioned into the current vectors $\bar{\imath}_b$ and $\bar{\jmath}_b$, which are associated with branches of the network, and the current vectors $\bar{\imath}_l$ and $\bar{\jmath}_l$, which are associated with links. The left side of equation (3.6.2) is

$$\left[\begin{array}{c} i_b + B_l^t i_l \\ \hline i_l \end{array}\right] + \left[\begin{array}{c} j_b + B_l^t j_l \\ \hline j_l \end{array}\right]$$

where

$\bar{\imath}_b + B_l^t \bar{\imath}_l = B^t \bar{\imath}$ and $\bar{\jmath}_b + B_l^t \bar{\jmath}_l = B^t \bar{\jmath}$

However

$B^t \bar{\imath} = 0$ and $B^t \bar{\jmath} = \bar{I}_{BR}$

Then the left side of equation (3.6.2) is

$$\left[\begin{array}{c} 0 \\ \hline i_l \end{array}\right] + \left[\begin{array}{c} I_{BR} \\ \hline j_l \end{array}\right] = \left[\begin{array}{c} I_{BR} \\ \hline i_l + j_l \end{array}\right]$$

Since each element of $\bar{\imath}_l$ is equal to a current source of a fictitious branch, $\bar{\imath}_l + \bar{\jmath}_l$ is a vector in which each element is equal to the algebraic sum of the source currents of a fictitious branch and its associated link. Therefore,

$$\hat{I}_{BR} = \left[\begin{array}{c} I_{BR} \\ \hline i_l + j_l \end{array}\right]$$

and equation (3.6.1) becomes

$$\hat{I}_{BR} = \hat{B}^t[y]\bar{v} \qquad (3.6.3)$$

Since the voltages across the fictitious branches are zero, the voltage vector of the augmented network is

$$\hat{E}_{BR} = \begin{bmatrix} E_{BR} \\ \hline 0 \end{bmatrix}$$

The voltages across the elements of the original network from equation (3.5.14) are

$$\bar{v} = B\bar{E}_{BR}$$

However

$$B\bar{E}_{BR} = \hat{B}\hat{E}_{BR}$$

then

$$\bar{v} = \hat{B}\hat{E}_{BR} \qquad (3.6.4)$$

Substituting from equation (3.6.4) into equation (3.6.3)

$$\hat{I}_{BR} = \hat{B}^t[y]\hat{B}\hat{E}_{BR} \qquad (3.6.5)$$

Since the performance equation of the augmented network is

$$\hat{I}_{BR} = \hat{Y}_{BR}\hat{E}_{BR} \qquad (3.6.6)$$

it follows from equations (3.6.5) and (3.6.6) that the admittance matrix of the augmented network is

$$\hat{Y}_{BR} = \hat{B}^t[y]\hat{B} \qquad (3.6.7)$$

Equation (3.6.7) can be written in the partitioned form

$$\left[\begin{array}{c|c} Y_1 & Y_2 \\ \hline Y_3 & Y_4 \end{array}\right] = \left[\begin{array}{c|c} U_b & B_l^t \\ \hline 0 & U_l \end{array}\right] \left[\begin{array}{c|c} y_{bb} & y_{bl} \\ \hline y_{lb} & y_{ll} \end{array}\right] \left[\begin{array}{c|c} U_b & 0 \\ \hline B_l & U_l \end{array}\right] \qquad (3.6.8)$$

where $[y_{bb}]$ = primitive admittance matrix of branches

$[y_{bl}] = [y_{lb}]^t$ = primitive admittance matrix whose elements are the mutual admittances between branches and links

$[y_{ll}]$ = primitive admittance matrix of links

It follows from equation (3.6.8) that

$$Y_1 = [y_{bb}] + B_l^t[y_{lb}] + [y_{bl}]B_l + B_l^t[y_{ll}]B_l \qquad (3.6.9)$$

Since

$$Y_{BK} = B^t[y]B$$

or

$$Y_{BK} = \begin{bmatrix} U_b & B_l^t \end{bmatrix} \begin{bmatrix} y_{bb} & y_{bl} \\ y_{lb} & y_{ll} \end{bmatrix} \begin{bmatrix} U_b \\ B_l \end{bmatrix}$$

then

$$Y_{BK} = [y_{bb}] + B_l^t[y_{lb}] + [y_{bl}]B_l + B_l^t[y_{ll}]B_l \qquad (3.6.10)$$

From equations (3.6.9) and (3.6.10), therefore,

$$Y_{BK} = Y_1$$

The branch impedance matrix can be obtained from

$$Z_{BK} = Y_1^{-1}$$

Loop impedance and loop admittance matrices

The loop impedance matrix Z_{LOOP} can be obtained also by using the augmented loop incidence matrix $\hat{C}$ to relate the variables and parameters of the primitive network to those of an augmented interconnected network. The augmented network is obtained by connecting a fictitious link in parallel with each branch of the original network. In order to preserve the performance of the interconnected network the impedance of each fictitious link is set to zero and its voltage source is set equal and opposite to the voltage across the associated branch, as shown in Fig. 3.9*a*. The current through a fictitious link is zero. Then an open loop can be interpreted as a loop containing a branch and a fictitious link as shown in Fig. 3.9*b*.

The performance equation of the augmented network in the loop frame of reference is

$$\hat{E}_{LOOP} = \hat{Z}_{LOOP}\hat{I}_{LOOP}$$

The matrix Z_{LOOP} will be obtained directly from the impedance matrix $\hat{Z}_{LOOP}$ of the augmented network.

The performance equation for the primitive network

$$\bar{v} + \bar{e} = [z]\bar{\imath}$$

is premultiplied by $\hat{C}^t$ to obtain

$$\hat{C}^t\bar{v} + \hat{C}^t\bar{e} = \hat{C}^t[z]\bar{\imath} \qquad (3.6.11)$$

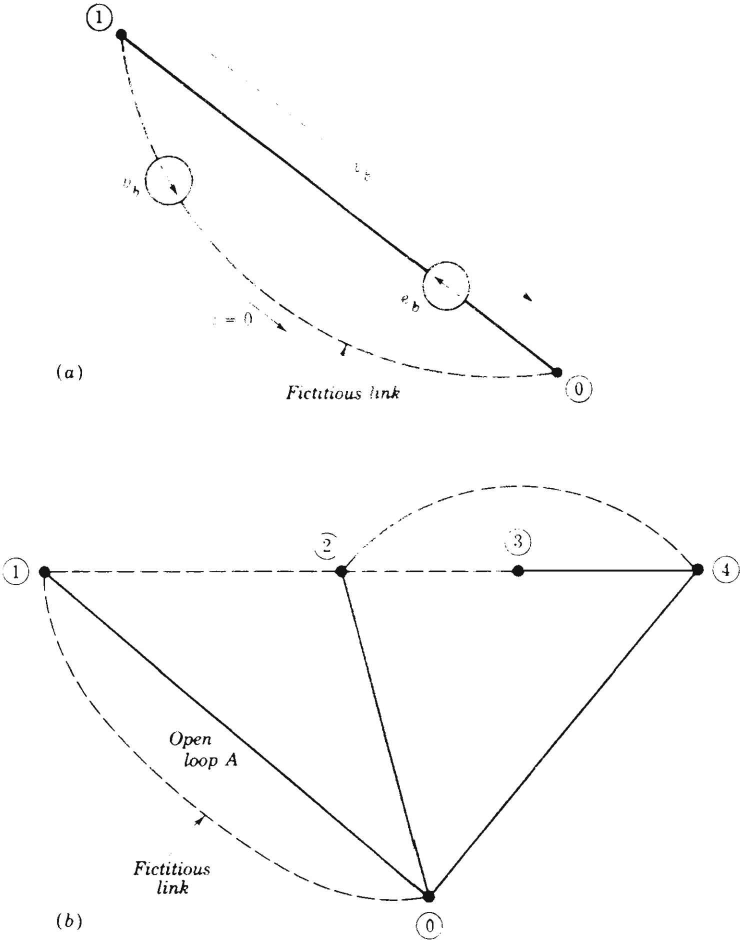


Fig. 3.9 **Representation of an augmented network.** (a) **Fictitious link in parallel with a branch;** (b) **interpretation of an open loop.**

Equation (3.6.11) can be written in the partitioned form

$$\left[\begin{array}{c|c} U_b & 0 \\ \hline C_b{}^t & U_l \end{array}\right] \left[\begin{array}{c} v_b \\ \hline v_l \end{array}\right] + \left[\begin{array}{c|c} U_b & 0 \\ \hline C_b{}^t & U_l \end{array}\right] \left[\begin{array}{c} e_b \\ \hline e_l \end{array}\right] = \left[\begin{array}{c|c} U_b & 0 \\ \hline C_b{}^t & U_l \end{array}\right] \left[\begin{array}{c} z \end{array}\right] \left[\begin{array}{c} i \end{array}\right] \tag{3.6.12}$$

where the primitive voltage vectors $\bar{v}$ and $\bar{e}$ are partitioned into the voltage vectors $\bar{v}_b$ and $\bar{e}_b$, which are associated with the branches of the network, and the voltage vectors $\bar{v}_l$ and $\bar{e}_l$, which are associated with the links. The left side of equation (3.6.12) is

$$\begin{array}{|c|}\hline v_b \\ \hline C_b{}^t v_b + v_l \\ \hline\end{array} + \begin{array}{|c|}\hline e_b \\ \hline C_b{}^t e_b + e_l \\ \hline\end{array}$$

where

$C_b{}^t\bar{v}_b + \bar{v}_l = C^t\bar{v}$ and $C_b{}^t\bar{e}_b + \bar{e}_l = C^t\bar{e}$

However

$C^t\bar{v} = 0$ and $C^t\bar{e} = \bar{E}_{LOOP}$

The left side of equation (3.6.12) is then

$$\begin{array}{|c|}\hline v_b \\ \hline 0 \\ \hline\end{array} + \begin{array}{|c|}\hline e_b \\ \hline E_{LOOP} \\ \hline\end{array} = \begin{array}{|c|}\hline v_b + e_b \\ \hline E_{LOOP} \\ \hline\end{array}$$

Since each element of $\bar{v}_b$ is equal to a voltage source of a fictitious link, $\bar{v}_b + \bar{e}_b$ is a vector in which each element is equal to the algebraic sum of the source voltages in an open loop. Therefore,

$$\hat{E}_{LOOP} = \begin{array}{|c|}\hline v_b + e_b \\ \hline E_{LOOP} \\ \hline\end{array} \tag{3.6.13}$$

and from equations (3.6.11) and (3.6.13)

$$\hat{E}_{LOOP} = \hat{C}^t[z]\bar{\imath} \tag{3.6.14}$$

Since the currents in the open loops are zero, the current vector of the augmented network is

$$\hat{I}_{LOOP} = \begin{array}{|c|}\hline 0 \\ \hline I_{LOOP} \\ \hline\end{array}$$

The currents through the elements of the original network from equation (3.5.20) are

$$\bar{\imath} = C\bar{I}_{LOOP}$$

However,

$$C\bar{I}_{LOOP} = \hat{C}\hat{I}_{LOOP}$$

then

$$\bar{I} = \hat{C}\hat{I}_{LOOP} \tag{3.6.15}$$

Substituting from equation (3.6.15) into equation (3.6.14),

$$\hat{E}_{LOOP} = \hat{C}^t[z]\hat{C}\hat{I}_{LOOP} \tag{3.6.16}$$

Since the performance equation of the augmented network is

$$\hat{E}_{LOOP} = \hat{Z}_{LOOP}\hat{I}_{LOOP} \tag{3.6.17}$$

it follows from equations (3.6.16) and (3.6.17) that the impedance matrix of the augmented network is

$$\hat{Z}_{LOOP} = \hat{C}^t[z]\hat{C} \tag{3.6.18}$$

Equation (3.6.18) can be written in the partitioned form:

$$\left[\begin{array}{c|c} Z_1 & Z_2 \\ \hline Z_3 & Z_4 \end{array}\right] = \left[\begin{array}{c|c} U_b & 0 \\ \hline C_b^t & U_l \end{array}\right] \left[\begin{array}{c|c} z_{bb} & z_{bl} \\ \hline z_{lb} & z_{ll} \end{array}\right] \left[\begin{array}{c|c} U_b & C_b \\ \hline 0 & U_l \end{array}\right] \tag{3.6.19}$$

where $[z_{bb}]$ = primitive impedance matrix of branches
$[z_{bl}]$ = $[z_{lb}]^t$ = primitive impedance matrix whose elements are the mutual impedances between branches and links
$[z_{ll}]$ = primitive impedance matrix of links

It follows from equation (3.6.19) that

$$Z_4 = C_b^t[z_{bb}]C_b + [z_{lb}]C_b + C_b^t[z_{bl}] + [z_{ll}] \tag{3.6.20}$$

Since

$$Z_{LOOP} = C^t[z]C$$

or

$$Z_{LOOP} = \left[\begin{array}{c|c} C_b^t & U_l \end{array}\right] \left[\begin{array}{c|c} z_{bb} & z_{bl} \\ \hline z_{lb} & z_{ll} \end{array}\right] \left[\begin{array}{c} C_b \\ \hline U_l \end{array}\right]$$

then

$$Z_{LOOP} = C_b^t[z_{bb}]C_b + [z_{lb}]C_b + C_b^t[z_{bl}] + [z_{ll}] \tag{3.6.21}$$

From equations (3.6.20) and (3.6.21), therefore,

$$Z_{LOOP} = Z_4$$

The loop admittance matrix can be obtained from

$Y_{LOOP} = Z_4^{-1}$

Derivation of loop admittance matrix from augmented network admittance matrix

The loop admittance matrix Y_{LOOP} can be obtained from the augmented admittance matrix $\hat{Y}_{BR}$. From equations (3.6.7) and (3.6.18),

$$\hat{Z}_{LOOP}\hat{Y}_{BR} = \hat{C}^t[z]\hat{C}\hat{B}^t[y]\hat{B} \tag{3.6.22}$$

In partitioned form.

$$\hat{C}\hat{B}^t = \left[\begin{array}{c|c} U_b & C_b \\ \hline 0 & U_l \end{array}\right] \left[\begin{array}{c|c} U_b & B_l^t \\ \hline 0 & U_l \end{array}\right] = \left[\begin{array}{c|c} U_b & B_l^t + C_b \\ \hline 0 & U_l \end{array}\right] \tag{3.6.23}$$

The currents through the elements of the primitive network from equation (3.5.20) are

$\bar{i} = C\bar{I}_{LOOP}$

Premultiplying by B^t,

$$B^t\bar{i} = B^tC\bar{I}_{LOOP} \tag{3.6.24}$$

However, from equation (3.5.11) the left side of equation (3.6.24) is zero. Therefore, equation (3.6.24) can be written

$(C_b + B_l^t)\bar{I}_{LOOP} = 0$

It follows that

$$C_b = -B_l^t \tag{3.6.25}$$

Substituting from equation (3.6.25) into equation (3.6.23),

$$\hat{C}\hat{B}^t = U \tag{3.6.26}$$

In a similar manner it can be shown that

$$\hat{C}^t\hat{B} = U \tag{3.6.27}$$

Substituting from equation (3.6.26) into (3.6.22),

$\hat{Z}_{LOOP}\hat{Y}_{BR} = \hat{C}^t[z][y]\hat{B}$

Since

$[z][y] = U$

then

$$\hat{Z}_{LOOP}\hat{Y}_{BR} = \hat{C}^t\hat{B}$$

Therefore, from equation (3.6.27),

$$\hat{Z}_{LOOP}\hat{Y}_{BR} = U \qquad (3.6.28)$$

Equation (3.6.28) in partitioned form is

$$\left[\begin{array}{cc} Z_1 & Z_2 \\ \hline Z_3 & Z_4 \end{array}\right]\left[\begin{array}{cc} Y_1 & Y_2 \\ \hline Y_3 & Y_4 \end{array}\right] = \left[\begin{array}{c|c} U_b & 0 \\ \hline 0 & U_l \end{array}\right]$$

It follows that

$$Z_1Y_1 + Z_2Y_3 = U_b \qquad (3.6.29)$$
$$Z_1Y_2 + Z_2Y_4 = 0$$
$$Z_3Y_1 + Z_4Y_3 = 0 \qquad (3.6.30)$$
$$Z_3Y_2 + Z_4Y_4 = U_l \qquad (3.6.31)$$

Solving for Z_3 from equation (3.6.30),

$$Z_3 = -Z_4Y_3Y_1^{-1}$$

and substituting into equation (3.6.31),

$$-Z_4Y_3Y_1^{-1}Y_2 + Z_4Y_4 = U_l$$

or

$$Z_4(Y_4 - Y_3Y_1^{-1}Y_2) = U_l$$

Since

$$Z_4Y_{LOOP} = U_l$$

it follows that

$$Y_{LOOP} = Y_4 - Y_3Y_1^{-1}Y_2$$

Derivation of branch impedance matrix from augmented impedance matrix

The branch impedance matrix Z_{BR} can be obtained from the augmented impedance matrix $\hat{Z}_{LOOP}$. Combining equations (3.6.29) and (3.6.30) yields

$$(Z_1 - Z_2Z_4^{-1}Z_3)Y_1 = U_b$$

Since

$$Z_{BR}Y_1 = U_b$$

it follows that

$$Z_{BR} = Z_1 - Z_2Z_4^{-1}Z_3$$

Derivation of branch admittance and impedance matrices from bus admittance and impedance matrices

Using the branch-path incidence matrix K the branch admittance matrix Y_{BR} can be obtained from Y_{BUS}. From equation (3.3.1),

$$A_bK^t = U_b$$

and from equation (3.3.3),

$$B_l = A_lK^t$$

Postmultiplying A by K^t,

$$AK^t = \begin{bmatrix} A_b \\ \hline A_l \end{bmatrix} K^t = \begin{bmatrix} A_bK^t \\ \hline A_lK^t \end{bmatrix} \tag{3.6.32}$$

Substituting from equations (3.3.1) and (3.3.3) into (3.6.32),

$$AK^t = \begin{bmatrix} U_b \\ \hline B_l \end{bmatrix} = B$$

Transposing,

$$KA^t = B^t$$

Postmultiplying by $[y]AK^t$ yields

$$KA^t[y]AK^t = B^t[y]AK^t$$

or

$$K(A^t[y]A)K^t = B^t[y]B \tag{3.6.33}$$

From the singular transformations,

$$Y_{BUS} = A^t[y]A \qquad \text{and} \qquad Y_{BR} = B^t[y]B$$

Hence equation (3.6.33) becomes

$$Y_{BR} = KY_{BUS}K^t \tag{3.6.34}$$

The branch impedance matrix is

$$Z_{BR} = Y_{BR}^{-1} = (K^t)^{-1}Y_{BUS}^{-1}K^{-1} \tag{3.6.35}$$

From equation (3.3.2),

$$K^t = A_b^{-1} \tag{3.6.36}$$

Substituting from equation (3.6.36) into equation (3.6.35),

$$Z_{BR} = A_bZ_{BUS}A_b^t$$

Derivation of bus admittance and impedance matrices from branch admittance and impedance matrices

Equation (3.6.34) is premultiplied by K^{-1} and postmultiplied by $(K^t)^{-1}$ to obtain

$$K^{-1}Y_{BR}(K^t)^{-1} = Y_{BUS} \tag{3.6.37}$$

Substituting from equation (3.6.36) into equation (3.6.37),

$$Y_{BUS} = A_b^tY_{BR}A_b$$

Since

$$Z_{BUS} = Y_{BUS}^{-1}$$

then

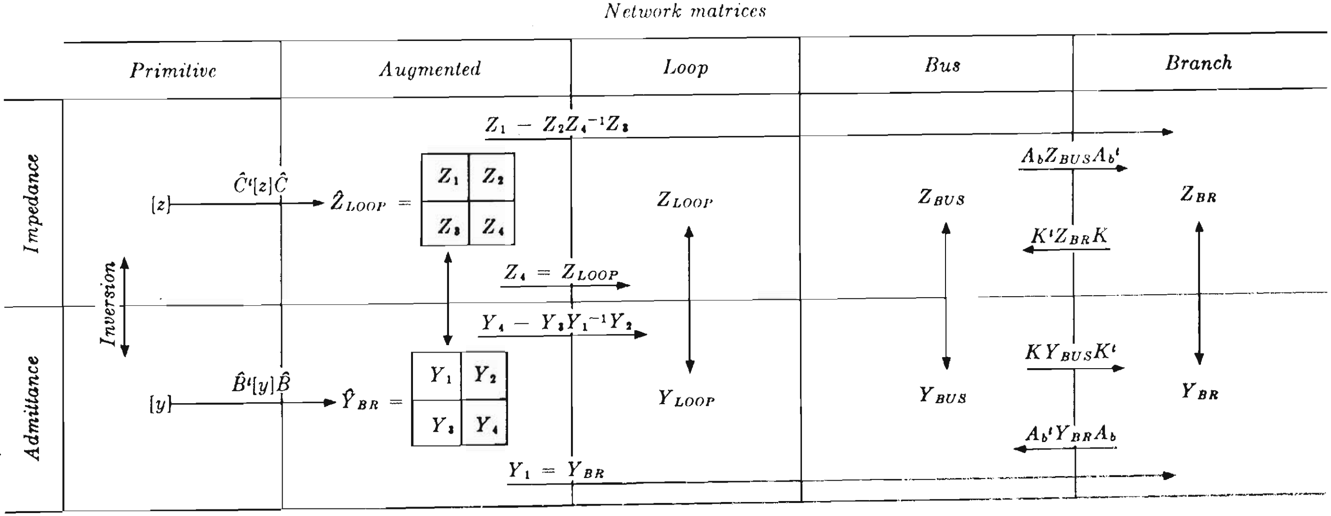
$$Z_{BUS} = (A_b^tY_{BR}A_b)^{-1} \qquad \text{or} \qquad Z_{BUS} = K^tZ_{BR}K$$

The nonsingular transformations for obtaining network matrices are summarized in Table 3.3.

3.7 Example of formation of incidence and network matrices

The method of forming the incidence and network matrices will be illustrated for the network shown in Fig. 3.10. The incidence matrices for a given network are not unique and depend on the orientation of the graph and the selection of branches, basic cut-sets, and basic loops. However, the network matrices are unique.

Table 3.3 Formation of network matrices by nonsingular transformations



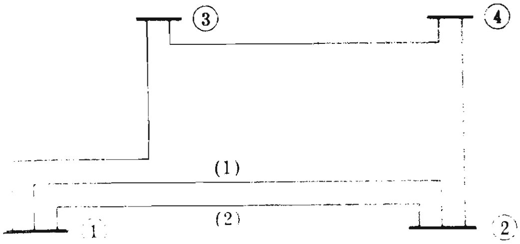


Fig. 3.10 Sample network.

Problem

a. Form the incidence matrices $\hat{A}$, A, K, B, $\hat{B}$, C, and $\hat{C}$ for the network shown in Fig. 3.10.
b. Form the network matrices Y_{BUS}, Y_{BR}, and Z_{LOOP} by singular transformations.
c. Form the network matrices Z_{LOOP}, Z_{BR}, and Z_{BUS} by nonsingular transformations.

Solution

The impedance data for the sample network is given in Table 3.4.

Table 3.4 Impedances for sample network

	Self		*Mutual*	
Element number	*Bus code* p–q	*Impedance* $z_{pq,pq}$	*Bus code* r–s	*Impedance* $z_{pq,rs}$
1	1–2(1)	0.6		
2	1–3	0.5	1–2(1)	0.1
3	3–4	0.5		
4	1–2(2)	0.4	1–2(1)	0.2
5	2–4	0.2		

The network contains four nodes and five elements, that is, $n = 4$ and $e = 5$. The number of branches is

$$b = n - 1 = 3$$

and the number of basic loops is

$$l = e - n + 1 = 2$$

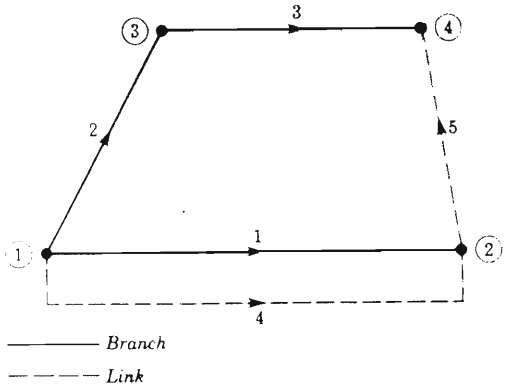


Fig. 3.11 Tree and cotree of the oriented connected graph of sample network.

a. The branches and links of the oriented connected graph of the network are shown in Fig. 3.11. The element-node incidence matrix is

$$\hat{A} = \begin{array}{c|cccc} e \backslash n & ① & ② & ③ & ④ \\ \hline 1 & 1 & -1 & & \\ 2 & 1 & & -1 & \\ 3 & & & 1 & -1 \\ 4 & 1 & -1 & & \\ 5 & & 1 & & -1 \end{array}$$

Selecting node 1 as the reference, the bus incidence matrix is

$$A = \begin{array}{c|ccc} e \backslash \text{bus} & ② & ③ & ④ \\ \hline 1 & -1 & & \\ 2 & & -1 & \\ 3 & & 1 & -1 \\ 4 & -1 & & \\ 5 & 1 & & -1 \end{array}$$

The branch-path incidence matrix is

$$K = \begin{array}{c|ccc} b \backslash \text{path} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline 1 & -1 & & \\ 2 & & -1 & -1 \\ 3 & & & -1 \end{array}$$

The basic and tie cut-sets of the oriented connected graph of the network are shown in Fig. 3.12. The basic cut-set incidence matrix is

$$B = \begin{array}{c|ccc} e \backslash b & A & B & C \\ \hline 1 & 1 & & \\ 2 & & 1 & \\ 3 & & & 1 \\ 4 & 1 & & \\ 5 & -1 & 1 & 1 \end{array}$$

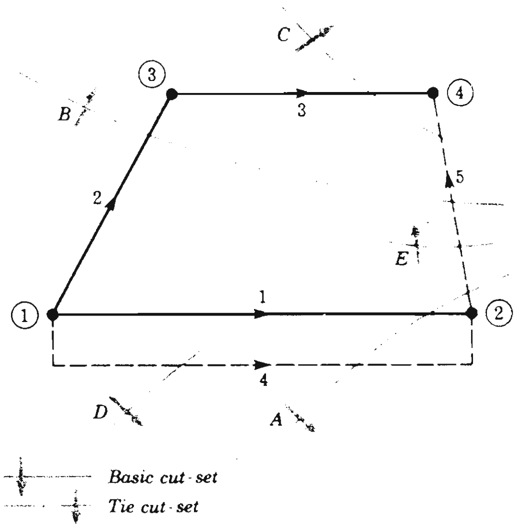


Fig. 3.12 Basic and tie cut-sets of the oriented connected graph of sample network.

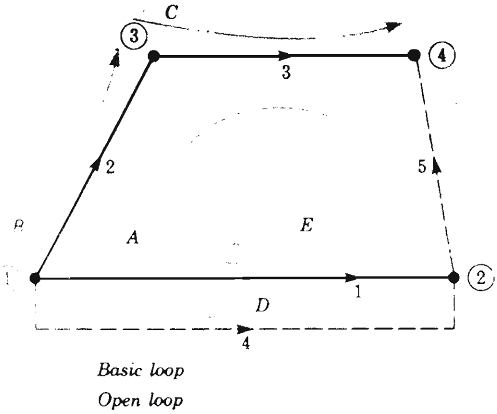


Fig. 3.13 Basic and open loops of the oriented connected graph of sample network.

The augmented cut-set incidence matrix is

$$\hat{B} = \begin{array}{c|ccccc} e \backslash e & A & B & C & D & E \\ \hline 1 & 1 & & & & \\ 2 & & 1 & & & \\ 3 & & & 1 & & \\ 4 & 1 & & & 1 & \\ 5 & -1 & 1 & 1 & & 1 \end{array}$$

The basic and open loops of the oriented connected graph are shown in Fig. 3.13. The basic loop incidence matrix is

$$C = \begin{array}{c|cc} {}_{e}\diagdown^{l} & D & E \\ \hline 1 & -1 & 1 \\ 2 & & -1 \\ 3 & & -1 \\ \hline 4 & 1 & \\ 5 & & 1 \end{array}$$

The augmented loop incidence matrix is

$$\hat{C} = \begin{array}{c|ccc|cc} {}_{e}\diagdown^{e} & A & B & C & D & E \\ \hline 1 & 1 & & & -1 & 1 \\ 2 & & 1 & & & -1 \\ 3 & & & 1 & & -1 \\ \hline 4 & & & & 1 & \\ 5 & & & & & 1 \end{array}$$

b. The primitive impedance matrix of the sample network from Table 3.4 is

$$[z] = \begin{array}{c|ccccc} {}_{e}\diagdown^{e} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0.6 & 0.1 & & 0.2 & \\ 2 & 0.1 & 0.5 & & & \\ 3 & & & 0.5 & & \\ 4 & 0.2 & & & 0.4 & \\ 5 & & & & & 0.2 \end{array}$$

By inversion, the primitive admittance matrix is

$$[y] = \begin{array}{c|ccccc} {}_{e}\diagdown^{e} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2.083 & -0.417 & & -1.042 & \\ 2 & -0.417 & 2.083 & & 0.208 & \\ 3 & & & 2.000 & & \\ 4 & -1.042 & 0.208 & & 3.021 & \\ 5 & & & & & 5.000 \end{array}$$

The bus admittance matrix obtained by a singular transformation is

$Y_{BUS} = A^t[y]A$

$$= \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline \textcircled{2} & -1 & & & -1 & 1 \\ \textcircled{3} & & -1 & 1 & & \\ \textcircled{4} & & & -1 & & -1 \end{array} \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2.083 & -0.417 & & -1.042 & \\ 2 & -0.417 & 2.083 & & 0.208 & \\ 3 & & & 2.000 & & \\ 4 & -1.042 & 0.208 & & 3.021 & \\ 5 & & & & & 5.000 \end{array} \begin{array}{c|ccc} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline 1 & -1 & & \\ 2 & & -1 & \\ 3 & & 1 & -1 \\ 4 & -1 & & \\ 5 & 1 & & -1 \end{array}$$

$$= \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline \textcircled{2} & -1.041 & 0.209 & & -1.979 & 5.000 \\ \textcircled{3} & 0.417 & -2.083 & 2.000 & -0.208 & \\ \textcircled{4} & & & -2.000 & & -5.000 \end{array} \begin{array}{c|ccc} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline 1 & -1 & & \\ 2 & & -1 & \\ 3 & & 1 & -1 \\ 4 & -1 & & \\ 5 & 1 & & -1 \end{array} = \begin{array}{c|ccc} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline \textcircled{2} & 8.020 & -0.209 & -5.000 \\ \textcircled{3} & -0.209 & 4.083 & -2.000 \\ \textcircled{4} & -5.000 & -2.000 & 7.000 \end{array}$$

The loop impedance matrix obtained by a singular transformation is

$Z_{LOOP} = C^t[z]C$

$$= \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline D & -1 & & & 1 & \\ \hline E & 1 & -1 & -1 & & 1 \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0.6 & 0.1 & & 0.2 & \\ \hline 2 & 0.1 & 0.5 & & & \\ \hline 3 & & & 0.5 & & \\ \hline 4 & 0.2 & & & 0.4 & \\ \hline 5 & & & & & 0.2 \\ \hline \end{array} \quad \begin{array}{c|c|c|} & D & E \\ \hline 1 & -1 & 1 \\ \hline 2 & & -1 \\ \hline 3 & & -1 \\ \hline 4 & 1 & \\ \hline 5 & & 1 \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline D & -0.4 & -0.1 & & 0.2 & \\ \hline E & 0.5 & -0.4 & -0.5 & 0.2 & 0.2 \\ \hline \end{array} \quad \begin{array}{c|c|c|} & D & E \\ \hline 1 & -1 & 1 \\ \hline 2 & & -1 \\ \hline 3 & & -1 \\ \hline 4 & 1 & \\ \hline 5 & & 1 \\ \hline \end{array} = \begin{array}{c|c|c|} & D & E \\ \hline D & 0.6 & -0.3 \\ \hline E & -0.3 & 1.6 \\ \hline \end{array}$$

The branch admittance matrix obtained by a singular transformation is

$$Y_{BR} = B^t[y]B$$

$$= \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 1 & & & 1 & -1 \\ B & & 1 & & & 1 \\ C & & & 1 & & 1 \\ \hline \end{array} \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2.083 & -0.417 & & -1.042 & \\ 2 & -0.417 & 2.083 & & 0.208 & \\ 3 & & & 2.000 & & \\ 4 & -1.042 & 0.208 & & 3.021 & \\ 5 & & & & & 5.000 \\ \hline \end{array} \begin{array}{c|ccc|} & A & B & C \\ \hline 1 & 1 & & \\ 2 & & 1 & \\ 3 & & & 1 \\ 4 & 1 & & \\ 5 & -1 & 1 & 1 \\ \hline \end{array}$$

$$= \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 1.041 & -0.209 & & 1.979 & -5.000 \\ B & -0.417 & 2.083 & & 0.208 & 5.000 \\ C & & & 2.000 & & 5.000 \\ \hline \end{array} \begin{array}{c|ccc|} & A & B & C \\ \hline 1 & 1 & & \\ 2 & & 1 & \\ 3 & & & 1 \\ 4 & 1 & & \\ 5 & -1 & 1 & 1 \\ \hline \end{array} = \begin{array}{c|ccc|} & A & B & C \\ \hline A & 8.020 & -5.209 & -5.000 \\ B & -5.209 & 7.083 & 5.000 \\ C & -5.000 & 5.000 & 7.000 \\ \hline \end{array}$$

c. The augmented network impedance matrix obtained by nonsingular transformation is

$$\hat{Z}_{LOOP} = \hat{C}^t[z]\hat{C} = \begin{array}{|c|c|}\hline Z_1 & Z_2 \\ \hline Z_3 & Z_4 \\ \hline\end{array}$$

$$= \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 1 & & & & \\ \hline B & & 1 & & & \\ \hline C & & & 1 & & \\ \hline D & -1 & & & 1 & \\ \hline E & 1 & -1 & -1 & & 1 \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & .6 & 1 & & .2 & \\ \hline 2 & .1 & .5 & & & \\ \hline 3 & & & .5 & & \\ \hline 4 & .2 & & & .4 & \\ \hline 5 & & & & & .2 \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|c|} & A & B & C & D & E \\ \hline 1 & 1 & & & -1 & 1 \\ \hline 2 & & 1 & & & -1 \\ \hline 3 & & & 1 & & -1 \\ \hline 4 & & & & 1 & \\ \hline 5 & & & & & 1 \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & .6 & .1 & & .2 & \\ \hline B & .1 & .5 & & & \\ \hline C & & & .5 & & \\ \hline D & -.4 & -.1 & & .2 & \\ \hline E & .5 & -.4 & -.5 & .2 & .2 \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|c|} & A & B & C & D & E \\ \hline 1 & 1 & & & -1 & 1 \\ \hline 2 & & 1 & & & -1 \\ \hline 3 & & & 1 & & -1 \\ \hline 4 & & & & 1 & \\ \hline 5 & & & & & 1 \\ \hline \end{array} = \begin{array}{c|c|c|c|c|c|} & A & B & C & D & E \\ \hline A & .6 & .1 & & -.4 & .5 \\ \hline B & .1 & .5 & & -.1 & -.4 \\ \hline C & & & .5 & & -.5 \\ \hline D & -.4 & -.1 & & .6 & -.3 \\ \hline E & .5 & -.4 & -.5 & -.3 & 1.6 \\ \hline \end{array}$$

Then the loop impedance matrix is

$$Z_{LOOP} = Z_4 = \begin{array}{c|c|c|} & D & E \\ \hline D & 0.6 & -0.3 \\ \hline E & -0.3 & 1.6 \\ \hline \end{array}$$

The branch impedance matrix is

$$Z_{BR} = Z_1 - Z_2 Z_4^{-1} Z_3$$

$$= \begin{array}{|c|c|c|}\hline 0.6 & 0.1 & \\ \hline 0.1 & 0.5 & \\ \hline & & 0.5 \\ \hline \end{array} - \begin{array}{|c|c|}\hline -0.4 & 0.5 \\ \hline -0.1 & -0.4 \\ \hline & -0.5 \\ \hline \end{array} \begin{array}{|c|c|}\hline 1.839 & 0.345 \\ \hline 0.345 & 0.690 \\ \hline \end{array} \begin{array}{|c|c|c|}\hline -0.4 & -0.1 & \\ \hline 0.5 & -0.4 & -0.5 \\ \hline \end{array}$$

$$= \begin{array}{|c|c|c|}\hline 0.6 & 0.1 & \\ \hline 0.1 & 0.5 & \\ \hline & & 0.5 \\ \hline \end{array} - \begin{array}{|c|c|c|}\hline 0.329 & -0.026 & -0.104 \\ \hline -0.026 & 0.156 & 0.155 \\ \hline -0.104 & 0.155 & 0.172 \\ \hline \end{array} = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & 0.271 & 0.126 & 0.104 \\ \hline 2 & 0.126 & 0.344 & -0.155 \\ \hline 3 & 0.104 & -0.155 & 0.328 \\ \hline \end{array}$$

The bus impedance matrix obtained by a nonsingular transformation is

$$Z_{BUS} = K^t Z_{BR} K$$

$$= \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline \textcircled{2} & -1 & & \\ \textcircled{3} & & -1 & \\ \textcircled{4} & & -1 & -1 \\ \hline \end{array} \quad \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 0.271 & 0.126 & 0.104 \\ 2 & 0.126 & 0.344 & -0.155 \\ 3 & 0.104 & -0.155 & 0.328 \\ \hline \end{array} \quad \begin{array}{c|ccc|} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline 1 & -1 & & \\ 2 & & -1 & -1 \\ 3 & & & -1 \\ \hline \end{array}$$

$$= \begin{array}{c|ccc|} & A & B & C \\ \hline \textcircled{2} & -0.271 & -0.126 & -0.104 \\ \textcircled{3} & -0.126 & -0.344 & 0.155 \\ \textcircled{4} & -0.230 & -0.189 & -0.173 \\ \hline \end{array} \quad \begin{array}{c|ccc|} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline 1 & -1 & & \\ 2 & & -1 & -1 \\ 3 & & & -1 \\ \hline \end{array} \quad = \quad \begin{array}{c|ccc|} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline \textcircled{2} & 0.271 & 0.126 & 0.230 \\ \textcircled{3} & 0.126 & 0.344 & 0.189 \\ \textcircled{4} & 0.230 & 0.189 & 0.362 \\ \hline \end{array}$$

Problems

3.1 Select for the sample network shown in Fig. 3.10 a different tree than that used in the example. Retain node 1 as the reference and form:

a. The incidence matrices $\hat{A}$, A, K, B, $\hat{B}$, C, and $\hat{C}$ and verify the following relations:

i. $A_bK^t = U$

ii. $B_l = A_lK^t$

iii. $C_b = -B_l^t$

iv. $\hat{C}\hat{B}^t = U$

b. The network matrices Y_{BUS}, Y_{BR}, and Z_{LOOP} by singular transformations

c. The network matrices Z_{LOOP}, Z_{BR}, and Z_{BUS} by nonsingular transformations

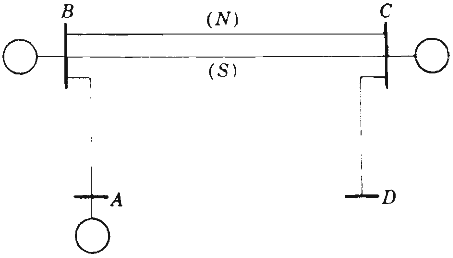


Fig. 3.14 Sample power system for Prob. 3.2.

3.2 The positive and zero sequence impedance data for the sample power system shown in Fig. 3.14 is given in Table 3.5. For this system:

a. Draw the positive sequence diagram and an oriented connected graph.

Table 3.5 Positive and zero sequence impedance data of sample power system for Prob. 3.2

Element	*Positive sequence impedance*	*Zero sequence impedance*	*Element*	*Mutual impedance*
Generator A	$0.0 + j0.25$	$0.0 + j0.1$		
Generator B	$0.0 + j0.25$	$0.0 + j0.1$		
Generator C	$0.0 + j0.25$	$0.0 + j0.1$		
Line A–B	$0.03 + j0.13$	$0.08 + j0.45$		
Line B–$C(N)$	$0.05 + j0.22$	$0.13 + j0.75$	Line B–$C(S)$	$0.08 + j0.48$
Line B–$C(S)$	$0.05 + j0.22$	$0.13 + j0.75$		
Line C–D	$0.02 + j0.11$	$0.07 + j0.37$		

b. Selecting ground as reference, form the incidence matrices A, K, B, $\hat{B}$, C, and $\hat{C}$ and verify the relations:
   i. $A_b K^t = U$
   ii. $B_l = A_l K^t$
   iii. $C_b = -B_l^t$
   iv. $\hat{C}\hat{B}^t = U$

c. Neglecting resistance, form the positive sequence network matrices Y_{BUS}, Z_{BUS}, Y_{BR}, Z_{BR}, Z_{LOOP} and Y_{LOOP} by singular transformations.

d. Neglecting resistance, form the zero sequence network matrices Y_{BUS}, Z_{BUS}, Y_{BR}, Z_{BR}, Z_{LOOP} and Y_{LOOP} by singular transformations.

e. Repeat *c* and *d* using nonsingular transformations.

f. Repeat *c* including resistance.

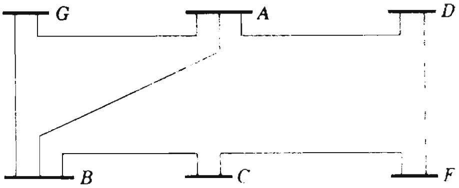


Fig. 3.15 Sample network for Prob. 3.3.

Table 3.6 Positive sequence reactances of sample network for Prob. 3.3

Element	*Positive sequence reactance*
G–A	0.04
G–B	0.05
A–B	0.04
B–C	0.03
A–D	0.02
C–F	0.07
D–F	0.10

3.3 The positive sequence reactances for the network shown in Fig. 3.15 are given in Table 3.6. Designate elements A–B and D–F as links and node G as the reference bus. Form:

a. The incidence matrices $\hat{A}$, A, K, B, $\hat{B}$, C, and $\hat{C}$

b. The network matrices Y_{BUS}, Y_{BR}, and Z_{LOOP} by singular transformations

c. The network matrices Y_{BUS}, Z_{BUS}, Z_{BR}, Z_{LOOP}, and Y_{LOOP} by nonsingular transformations

3.4 Prove that when there is no mutual coupling the diagonal and off-diagonal elements of the bus admittance matrix Y_{BUS} can be computed from

$$Y_{ii} = \sum_{j} y_{ij}$$

$$Y_{ij} = -y_{ij}$$

where y_{ij} is the sum of the admittances of all lines connecting buses i and j.

3.5 Using the bus impedance matrix Z_{BUS} computed in Prob. 3.2 and the internal generator voltages given in Table 3.7:

a. Compute the positive and zero sequence bus voltages of the network.

b. Compute the positive and zero sequence currents flowing in the line B–$C(N)$.

Table 3.7 Internal generator voltages for Prob. 3.5

	Internal per unit voltages	
Generator	*Positive sequence*	*Zero sequence*
A	$1.0\underline{/0°}$	0
B	$1.1\underline{/-10°}$	0
C	$1.0\underline{/-10°}$	$0.1\underline{/0°}$

3.6 Using the relations between interconnected and primitive network variables prove the following:

a. $A_b K^t = U$

b. $B_l = A_l K^t$

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chapter 4

Algorithms for formation of network matrices

4.1 Introduction

The methods presented in Secs. 3.5 and 3.6 require transformation and inversion of matrices to obtain network matrices. An alternative method based on an algorithm can be used to form the bus impedance matrix directly from system parameters and coded bus numbers. The underlying principle of the algorithm is the formation of the bus impedance matrix in steps, simulating the construction of the network by adding one element at a time (Brown, Person, Kirchmayer, and Stagg, 1960)†. A matrix is formed for the partial network represented after each element is connected to the network.

In addition, an algorithm is presented for deriving the loop admittance matrix from a given bus impedance matrix.

4.2 Algorithm for formation of bus impedance matrix

Performance equation of a partial network

Assume that the bus impedance matrix Z_{BUS} is known for a partial network of m buses and a reference node 0. The performance equation of this network, shown in Fig. 4.1, is

$$\bar{E}_{BUS} = Z_{BUS}\bar{I}_{BUS}$$

---

† Names in parentheses refer to the Bibliography at the end of each chapter.

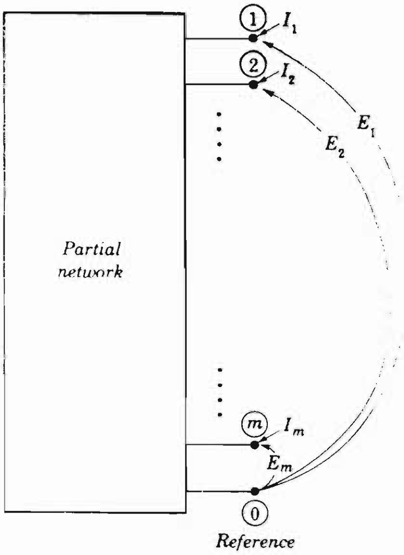

Fig. 4.1 **Representation of a partial network.**

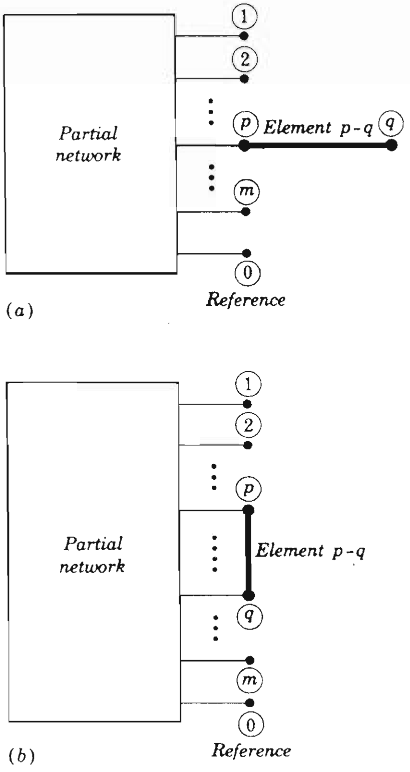

Fig. 4.2 **Representations of a partial network with an added element. (a) Addition of a branch; (b) addition of a link.**

where $\bar{E}_{BUS}$ = an $m \times 1$ vector of bus voltages measured with respect to the reference node

$\bar{I}_{BUS}$ = an $m \times 1$ vector of impressed bus currents

When an element p–q is added to the partial network it may be a branch or a link as shown in Fig. 4.2.

If p–q is a branch, a new bus q is added to the partial network and the resultant bus impedance matrix is of dimension $(m + 1) \times (m + 1)$. The new voltage and current vectors are of dimension $(m + 1) \times 1$. To determine the new bus impedance matrix requires only the calculation of the elements in the new row and column.

If p–q is a link, no new bus is added to the partial network. In this case, the dimensions of the matrices in the performance equation are unchanged, but all the elements of the bus impedance matrix must be recalculated to include the effect of the added link.

Addition of a branch

The performance equation for the partial network with an added branch p–q is

$$
\begin{array}{ccc|cccccccc|c}
 & & & 1 & & & p & & m & q & \\
E_1 & & 1 & Z_{11} & Z_{12} & \cdots & Z_{1p} & \cdots & Z_{1m} & Z_{1q} & I_1 \\
E_2 & & & Z_{21} & Z_{22} & \cdots & Z_{2p} & \cdots & Z_{2m} & Z_{2q} & I_2 \\
\cdots & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
E_p & = & p & Z_{p1} & Z_{p2} & \cdots & Z_{pp} & \cdots & Z_{pm} & Z_{pq} & I_p \\
\cdots & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
E_m & & m & Z_{m1} & Z_{m2} & \cdots & Z_{mp} & \cdots & Z_{mm} & Z_{mq} & I_m \\
E_q & & q & Z_{q1} & Z_{q2} & \cdots & Z_{qp} & \cdots & Z_{qm} & Z_{qq} & I_q
\end{array}
\qquad (4.2.1)
$$

It is assumed that the network consists of bilateral passive elements. Hence $Z_{qi} = Z_{iq}$ where $i = 1, 2, \ldots, m$ and refers to the buses of the partial network, not including the new bus q. The added branch p–q is assumed to be mutually coupled with one or more elements of the partial network.

The elements Z_{qi} can be determined by injecting a current at the ith bus and calculating the voltage at the qth bus with respect to the reference

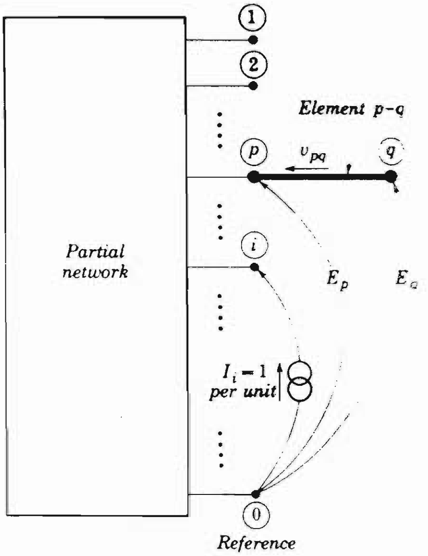

Fig. 4.3 Injected current and bus voltages for calculation of Z_{qi}.

node as shown in Fig. 4.3. Since all other bus currents equal zero, it follows from equation (4.2.1) that

$$
\begin{aligned}
E_1 &= Z_{1i}I_i \\
E_2 &= Z_{2i}I_i \\
&\cdots\cdots \\
E_p &= Z_{pi}I_i \\
&\cdots\cdots \\
E_m &= Z_{mi}I_i \\
E_q &= Z_{qi}I_i
\end{aligned}
\qquad (4.2.2)
$$

Letting $I_i = 1$ per unit in equations (4.2.2), Z_{qi} can be obtained directly by calculating E_q.

The bus voltages associated with the added element and the voltage across the element are related by

$$E_q = E_p - v_{pq} \qquad (4.2.3)$$

The currents in the elements of the network in Fig. 4.3 are expressed in terms of the primitive admittances and the voltages across the elements by

$$
\left[\begin{array}{c} i_{pq} \\ \hline i_{\rho\sigma} \end{array}\right]
=
\left[\begin{array}{c|c} y_{pq,pq} & y_{pq,\rho\sigma} \\ \hline y_{\rho\sigma,pq} & y_{\rho\sigma,\rho\sigma} \end{array}\right]
\left[\begin{array}{c} v_{pq} \\ \hline v_{\rho\sigma} \end{array}\right]
\qquad (4.2.4)
$$

In equation (4.2.4) pq is a fixed subscript and refers to the added element and $p\sigma$ is a variable subscript and refers to all other elements. Then,

i_{pq} and v_{pq}	are, respectively, current through and voltage across the added element
$\bar{i}_{p\sigma}$ and $\bar{v}_{p\sigma}$	are the current and voltage vectors of the elements of the partial network
$y_{pq,pq}$	is the self-admittance of the added element
$\bar{y}_{pq,p\sigma}$	is the vector of mutual admittances between the added element p–q and the elements p–σ of the partial network
$\bar{y}_{p\sigma,pq}$	is the transpose of the vector $\bar{y}_{pq,p\sigma}$
$[y_{p\sigma,p\sigma}]$	is the primitive admittance matrix of the partial network

The current in the added branch, shown in Fig. 4.3, is

$$i_{pq} = 0 \tag{4.2.5}$$

However v_{pq} is not equal to zero since the added branch is mutually coupled to one or more of the elements of the partial network. Moreover,

$$\bar{v}_{p\sigma} = \bar{E}_p - \bar{E}_\sigma \tag{4.2.6}$$

where $\bar{E}_p$ and $\bar{E}_\sigma$ are the voltages at the buses in the partial network. From equations (4.2.4) and (4.2.5),

$$i_{pq} = y_{pq,pq}v_{pq} + \bar{y}_{pq,p\sigma}\bar{v}_{p\sigma} = 0$$

and therefore,

$$v_{pq} = -\frac{\bar{y}_{pq,p\sigma}\bar{v}_{p\sigma}}{y_{pq,pq}}$$

Substituting for $\bar{v}_{p\sigma}$ from equation (4.2.6),

$$v_{pq} = -\frac{\bar{y}_{pq,p\sigma}(\bar{E}_p - \bar{E}_\sigma)}{y_{pq,pq}} \tag{4.2.7}$$

Substituting for v_{pq} in equation (4.2.3) from (4.2.7),

$$E_q = E_p + \frac{\bar{y}_{pq,p\sigma}(\bar{E}_p - \bar{E}_\sigma)}{y_{pq,pq}}$$

Finally, substituting for E_q, E_p, $\bar{E}_p$, and $\bar{E}_\sigma$ from equation (4.2.2) with $I_i = 1$,

$$Z_{qi} = Z_{pi} + \frac{\bar{y}_{pq,p\sigma}(\bar{Z}_{pi} - \bar{Z}_{\sigma i})}{y_{pq,pq}} \qquad \begin{matrix} i = 1, 2, \ldots, m \\ i \neq q \end{matrix} \tag{4.2.8}$$

The element Z_{qq} can be calculated by injecting a current at the qth bus and calculating the voltage at that bus. Since all other bus currents equal zero, it follows from equation (4.2.1) that

$$\begin{aligned}
E_1 &= Z_{1q}I_q \\
E_2 &= Z_{2q}I_q \\
&\cdots\cdots \\
E_p &= Z_{pq}I_q \\
&\cdots\cdots \\
E_m &= Z_{mq}I_q \\
E_q &= Z_{qq}I_q
\end{aligned} \tag{4.2.9}$$

Letting $I_q = 1$ per unit in equations (4.2.9), Z_{qq} can be obtained directly by calculating E_q.

The voltages at buses p and q are related by equation (4.2.3), and the current through the added element is

$$i_{pq} = -I_q = -1 \tag{4.2.10}$$

The voltages across the elements of the partial network are given by equation (4.2.6) and the currents through these elements by (4.2.4). From equations (4.2.4) and (4.2.10),

$$i_{pq} = y_{pq,pq}v_{pq} + \bar{y}_{pq,\rho\sigma}\bar{v}_{\rho\sigma} = -1$$

and therefore,

$$v_{pq} = -\frac{1 + \bar{y}_{pq,\rho\sigma}\bar{v}_{\rho\sigma}}{y_{pq,pq}}$$

Substituting for $\bar{v}_{\rho\sigma}$ from equation (4.2.6),

$$v_{pq} = -\frac{1 + \bar{y}_{pq,\rho\sigma}(\bar{E}_\rho - \bar{E}_\sigma)}{y_{pq,pq}} \tag{4.2.11}$$

Substituting for v_{pq} in equation (4.2.3) from (4.2.11),

$$E_q = E_p + \frac{1 + \bar{y}_{pq,\rho\sigma}(\bar{E}_\rho - \bar{E}_\sigma)}{y_{pq,pq}}$$

Finally, substituting for E_q, E_p, $\bar{E}_\rho$, and $\bar{E}_\sigma$ from equation (4.2.9) with $I_q = 1$,

$$Z_{qq} = Z_{pq} + \frac{1 + \bar{y}_{pq,\rho\sigma}(\bar{Z}_{\rho q} - \bar{Z}_{\sigma q})}{y_{pq,pq}} \tag{4.2.12}$$

If there is no mutual coupling between the added branch and other elements of the partial network, then the elements of $\bar{y}_{pq,\rho\sigma}$ are zero and

$$z_{pq,pq} = \frac{1}{y_{pq,pq}}$$

It follows from equation (4.2.8) that

$$Z_{qi} = Z_{pi} \qquad \begin{array}{l} i = 1, 2, \ldots, m \\ i \neq q \end{array}$$

and from equation (4.2.12) that

$$Z_{qq} = Z_{pq} + z_{pq,pq}$$

Furthermore, if there is no mutual coupling and p is the reference node,

$$Z_{pi} = 0 \qquad \begin{array}{l} i = 1, 2, \ldots, m \\ i \neq q \end{array}$$

and

$$Z_{qi} = 0 \qquad \begin{array}{l} i = 1, 2, \ldots, m \\ i \neq q \end{array}$$

Also

$$Z_{pq} = 0$$

and therefore,

$$Z_{qq} = z_{pq,pq}$$

Addition of a link

If the added element p–q is a link, the procedure for recalculating the elements of the bus impedance matrix is to connect in series with the added element a voltage source e_l as shown in Fig. 4.4. This creates a

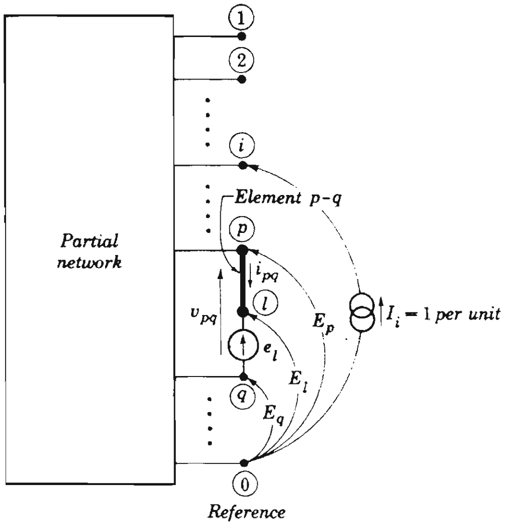

Fig. 4.4 Injected current voltage source in series with added link and bus voltages for calculation of Z_{li}.

fictitious node l which will be eliminated later. The voltage source e_l is selected such that the current through the added link is zero.

The performance equation for the partial network with the added element p–l and the series voltage source e_l is

$$\begin{array}{c} \\ E_1 \\ E_2 \\ \cdots \\ E_p \\ \cdots \\ E_m \\ e_l \end{array} \begin{array}{c} \\ \\ \\ \\ = \\ \\ \\ \\ \end{array} \begin{array}{c|ccccccc} & 1 & & & p & & m & l \\ 1 & Z_{11} & Z_{12} & \cdots & Z_{1p} & \cdots & Z_{1m} & Z_{1l} \\ & Z_{21} & Z_{22} & \cdots & Z_{2p} & \cdots & Z_{2m} & Z_{2l} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p & Z_{p1} & Z_{p2} & \cdots & Z_{pp} & \cdots & Z_{pm} & Z_{pl} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m & Z_{m1} & Z_{m2} & \cdots & Z_{mp} & \cdots & Z_{mm} & Z_{ml} \\ l & Z_{l1} & Z_{l2} & \cdots & Z_{lp} & \cdots & Z_{lm} & Z_{ll} \end{array} \begin{array}{c} \\ I_1 \\ I_2 \\ \cdots \\ I_p \\ \cdots \\ I_m \\ I_l \end{array} \qquad (4.2.13)$$

Since

$$e_l = E_l - E_q$$

the element Z_{li} can be determined by injecting a current at the ith bus and calculating the voltage at the lth node with respect to bus q. Since all other bus currents equal zero, it follows from equation (4.2.13) that

$$\begin{aligned} E_k &= Z_{ki} I_i \qquad k = 1, 2, \ldots, m \\ e_l &= Z_{li} I_i \end{aligned} \qquad (4.2.14)$$

Letting $I_i = 1$ per unit in equations (4.2.14), Z_{li} can be obtained directly by calculating e_l.

The series voltage source is

$$e_l = E_p - E_q - v_{pl} \qquad (4.2.15)$$

Since the current through the added link is

$$i_{pq} = 0$$

the element p–l can be treated as a branch. The current in this element in terms of primitive admittances and the voltages across the elements is

$$i_{pl} = y_{pl,pl} v_{pl} + \bar{y}_{pl,\rho\sigma} \bar{v}_{\rho\sigma}$$

where

$$i_{pi} = i_{pq} = 0$$

Therefore

$$v_{pl} = -\frac{\bar{y}_{pl,\rho\sigma}\bar{v}_{\rho\sigma}}{y_{pl,pl}}$$

Since

$$\bar{y}_{pl,\rho\sigma} = \bar{y}_{pq,\rho\sigma} \qquad \text{and} \qquad y_{pl,pl} = y_{pq,pq}$$

then

$$v_{pl} = -\frac{\bar{y}_{pq,\rho\sigma}\bar{v}_{\rho\sigma}}{y_{pq,pq}} \tag{4.2.16}$$

Substituting in order from equations (4.2.16), (4.2.6), and (4.2.14) with $I_i = 1$ into equation (4.2.15) yields

$$Z_{li} = Z_{pi} - Z_{qi} + \frac{\bar{y}_{pq,\rho\sigma}(\bar{Z}_{\rho i} - \bar{Z}_{\sigma i})}{y_{pq,pq}} \qquad \begin{matrix} i = 1, 2, \ldots, m \\ i \neq l \end{matrix} \tag{4.2.17}$$

The element Z_{ll} can be calculated by injecting a current at the lth bus with bus q as reference and calculating the voltage at the lth bus with respect to bus q. Since all other bus currents equal zero, it follows from equation (4.2.13) that

$$\begin{aligned} E_k &= Z_{kl}I_l \qquad k = 1, 2, \ldots, m \\ e_l &= Z_{ll}I_l \end{aligned} \tag{4.2.18}$$

Letting $I_l = 1$ per unit in equation (4.2.18), Z_{ll} can be obtained directly by calculating e_l.

The current in the element p–l is

$$i_{pl} = -I_l = -1$$

This current in terms of primitive admittances and the voltages across the elements is

$$i_{pl} = y_{pl,pl}v_{pl} + \bar{y}_{pl,\rho\sigma}\bar{v}_{\rho\sigma} = -1$$

Again, since

$$\bar{y}_{pl,\rho\sigma} = \bar{y}_{pq,\rho\sigma} \qquad \text{and} \qquad y_{pl,pl} = y_{pq,pq}$$

then

$$v_{pl} = -\frac{1 + \bar{y}_{pq,\rho\sigma}\bar{v}_{\rho\sigma}}{y_{pq,pq}} \tag{4.2.19}$$

Substituting in order from equations (4.2.19), (4.2.6), and (4.2.18) with $I_l = 1$ into (4.2.15) yields

$$Z_{ll} = Z_{pl} - Z_{ql} + \frac{1 + \bar{y}_{pq,\rho\sigma}(\bar{Z}_{\rho l} - \bar{Z}_{\sigma l})}{y_{pq,pq}} \tag{4.2.20}$$

If there is no mutual coupling between the added element and other elements of the partial network, the elements of $\bar{y}_{pq,\rho\sigma}$ are zero and

$$z_{pq,pq} = \frac{1}{y_{pq,pq}}$$

It follows from equation (4.2.17) that

$$Z_{li} = Z_{pi} - Z_{qi} \qquad \begin{matrix} i = 1, 2, \ldots, m \\ i \neq l \end{matrix}$$

and from equation (4.2.20),

$$Z_{ll} = Z_{pl} - Z_{ql} + z_{pq,pq}$$

Furthermore, if there is no mutual coupling and p is the reference node,

$$Z_{pi} = 0 \qquad \begin{matrix} i = 1, 2, \ldots, m \\ i \neq l \end{matrix}$$

and

$$Z_{li} = -Z_{qi} \qquad \begin{matrix} i = 1, 2, \ldots, m \\ i \neq l \end{matrix}$$

Also

$$Z_{pl} = 0$$

and therefore,

$$Z_{ll} = -Z_{ql} + z_{pq,pq}$$

The elements in the lth row and column of the bus impedance matrix for the augmented partial network are found from equations (4.2.17) and (4.2.20). It remains to calculate the required bus impedance matrix to include the effect of the added link. This can be accomplished by modifying the elements Z_{ij}, where $i, j = 1, 2, \ldots, m$, and eliminating the lth row and column corresponding to the fictitious node.

The fictitious node l is eliminated by short circuiting the series voltage source e_l. From equation (4.2.13),

$$\bar{E}_{BUS} = Z_{BUS}\bar{I}_{BUS} + \bar{Z}_{il}I_l \tag{4.2.21}$$

and

$$e_l = \bar{Z}_{lj}\bar{I}_{BUS} + Z_{ll}I_l = 0 \tag{4.2.22}$$

where $i, j = 1, 2, \ldots, m$. Solving for I_l from equation (4.2.22) and substituting into (4.2.21),

$$\bar{E}_{BUS} = \left(Z_{BUS} - \frac{\bar{Z}_{il}\bar{Z}_{lj}}{Z_{ll}} \right) \bar{I}_{BUS}$$

which is the performance equation of the partial network including the link p–q. It follows that the required bus impedance matrix is

$$Z_{BUS\,(\text{modified})} = Z_{BUS\,(\text{before elimination})} - \frac{\bar{Z}_{il}\bar{Z}_{lj}}{Z_{ll}}$$

where any element of $Z_{BUS\,(\text{modified})}$ is

$$Z_{ij\,(\text{modified})} = Z_{ij\,(\text{before elimination})} - \frac{Z_{il}Z_{lj}}{Z_{ll}}$$

A summary of the equations for the formation of the bus impedance matrix is given in Table 4.1.

4.3 Modification of the bus impedance matrix for changes in the network

The bus impedance matrix Z_{BUS} can be modified to reflect changes in the network. These changes may be addition of elements, removal of elements, or changes in the impedances of elements.

The method described in Sec. 4.2 based on the algorithm for forming a bus impedance matrix can be applied if elements are added to the network. Then Z_{BUS} is considered the matrix of the partial network at that stage and the new elements are added one at a time to produce the new bus impedance matrix Z'_{BUS}.

The procedure to remove elements or to change the impedances of elements is the same. If an element is removed which is not mutually coupled to any other element, the modified bus impedance matrix can be obtained by adding, in parallel with the element, a link whose impedance is equal to the negative of the impedance of the element to be removed. If the impedance of an uncoupled element is changed, the modified bus impedance matrix can be obtained by adding a link in parallel with the element such that the equivalent impedance of the two elements is the desired value.

When mutually coupled elements are removed or their impedances changed, the modified bus impedance matrix can not be obtained by adding a link and using the procedure described in Sec. 4.2. However, an equation can be derived for modifying the elements of Z_{BUS} by introducing appropriate changes in the bus currents of the original net-

Table 4.1 **Summary of equations for formation of bus impedance matrix**

Add p q	*Mutual coupling*		*No mutual coupling*	
	p is not the reference bus	*p is the reference bus*	*p is not the reference bus*	*p is the reference bus*
Branch	$Z_{qi} = Z_{pi} + \frac{\bar{y}_{pq,p\sigma}(\bar{Z}_{pi} - \bar{Z}_{\sigma i})}{y_{pq,pq}}$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{qq} = Z_{pq} + \frac{1 + \bar{y}_{pq,p\sigma}(\bar{Z}_{pq} - \bar{Z}_{\sigma q})}{y_{pq,pq}}$	$Z_{qi} = \frac{\bar{y}_{pq,p\sigma}(\bar{Z}_{pi} - \bar{Z}_{\sigma i})}{y_{pq,pq}}$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{qq} = \frac{1 + \bar{y}_{pq,p\sigma}(\bar{Z}_{pq} - \bar{Z}_{\sigma q})}{y_{pq,pq}}$	$Z_{qi} = Z_{pi}$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{qq} = Z_{pq} + z_{pq,pq}$	$Z_{qi} = 0$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{qq} = z_{pq,pq}$
Link	$Z_{li} = Z_{pi} - Z_{qi} + \frac{\bar{y}_{pq,p\sigma}(\bar{Z}_{pi} - \bar{Z}_{\sigma i})}{y_{pq,pq}}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{ll} = Z_{pl} - Z_{ql} + \frac{1 + \bar{y}_{pq,p\sigma}(\bar{Z}_{pl} - \bar{Z}_{\sigma l})}{y_{pq,pq}}$	$Z_{li} = -Z_{qi} + \frac{\bar{y}_{pq,p\sigma}(\bar{Z}_{pi} - \bar{Z}_{\sigma i})}{y_{pq,pq}}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{ll} = -Z_{ql} + \frac{1 + \bar{y}_{pq,p\sigma}(\bar{Z}_{pl} - \bar{Z}_{\sigma l})}{y_{pq,pq}}$	$Z_{li} = Z_{pi} - Z_{qi}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{ll} = Z_{pl} - Z_{ql} + z_{pq,pq}$	$Z_{li} = -Z_{qi}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{ll} = -Z_{ql} + z_{pq,pq}$

Modification of the elements for elimination of lth node

$$Z_{ij\,(\text{modified})} = Z_{ij\,(\text{before elimination})} - \frac{Z_{il}Z_{lj}}{Z_{ll}} \qquad i, j = 1, 2, \ldots, m$$

work to simulate the removal of elements or changes in their impedances. The performance equation in terms of the new bus currents is

$$\bar{E}'_{BUS} = Z_{BUS}(\bar{I}_{BUS} + \overline{\Delta I}_{BUS}) \tag{4.3.1}$$

where $\overline{\Delta I}_{BUS}$ is a vector of bus current changes such that $\bar{E}'_{BUS}$ will reflect the desired changes in the network.

An element Z'_{ij} of the modified bus impedance matrix can be obtained by calculating for the modified network the voltage at bus i with a current injected at bus j. This is equivalent to calculating for the original network the voltage at bus i with the same value of current injected at bus j and appropriate changes in currents at the buses which are terminals of the elements being changed.

If the elements μ-ν coupled to elements ρ-σ are removed or their impedances are changed, the corresponding changes in the bus currents are

$$\begin{aligned}
\Delta I_k &= \Delta i_{\mu\nu} & k &= \mu \\
\Delta I_k &= -\Delta i_{\mu\nu} & k &= \nu \\
\Delta I_k &= \Delta i_{\rho\sigma} & k &= \rho \\
\Delta I_k &= -\Delta i_{\rho\sigma} & k &= \sigma \\
\Delta I_k &= 0 & &\text{for all other } k
\end{aligned} \tag{4.3.2}$$

Letting the injected current at the jth bus equal one per unit,

$$\begin{aligned}
I_j &= 1 \\
I_k &= 0 \qquad \begin{matrix} k = 1, 2, \ldots, n \\ k \neq j \end{matrix}
\end{aligned} \tag{4.3.3}$$

From the performance equation (4.3.1),

$$E'_i = \sum_{k=1}^{n} Z_{ik}(I_k + \Delta I_k) \qquad i = 1, 2, \ldots, n$$

Substituting for ΔI_k and I_k from equations (4.3.2) and (4.3.3),

$$E'_i = Z_{ij} + \bar{Z}_{i\mu}\overline{\Delta i}_{\mu\nu} - \bar{Z}_{i\nu}\overline{\Delta i}_{\mu\nu} + \bar{Z}_{i\rho}\overline{\Delta i}_{\rho\sigma} - \bar{Z}_{i\sigma}\overline{\Delta i}_{\rho\sigma}$$

$$E'_i = Z_{ij} + (\bar{Z}_{i\mu} - \bar{Z}_{i\nu})\overline{\Delta i}_{\mu\nu} + (\bar{Z}_{i\rho} - \bar{Z}_{i\sigma})\overline{\Delta i}_{\rho\sigma}$$

Using the subscript $\alpha\beta$ for network elements μ-ν and ρ-σ,

$$E'_i = Z_{ij} + (\bar{Z}_{i\alpha} - \bar{Z}_{i\beta})\overline{\Delta i}_{\alpha\beta} \qquad i = 1, 2, \ldots, n \tag{4.3.4}$$

From the performance equation of the primitive network,

$$\overline{\Delta i}_{\alpha\beta} = ([y_s] - [y'_s])\bar{v}_{\gamma\delta} \tag{4.3.5}$$

where $[y_s]$ and $[y'_s]$ are respectively the square submatrices of the original and modified primitive admittance matrices. The rows and columns of the submatrices correspond to the network elements μ–ν and ρ–σ. The subscripts of the elements of $([y_s] - [y'_s])$ are $\alpha\beta$, $\gamma\delta$. The voltage vector in equation (4.3.5) is

$$\bar{v}'_{\gamma\delta} = \bar{E}'_{\gamma} - \bar{E}'_{\delta}$$

Substituting for $\bar{E}'_{\gamma}$ and $\bar{E}'_{\delta}$ from equation (4.3.4),

$$\bar{v}'_{\gamma\delta} = \bar{Z}_{\gamma j} - \bar{Z}_{\delta j} + ([Z_{\gamma\alpha}] - [Z_{\delta\alpha}] - [Z_{\gamma\beta}] + [Z_{\delta\beta}])\overline{\Delta i}_{\alpha\beta} \qquad (4.3.6)$$

Substituting from equation (4.3.6) for $\bar{v}'_{\gamma\delta}$ into (4.3.5),

$$\overline{\Delta i}_{\alpha\beta} = ([y_s] - [y'_s])\{\bar{Z}_{\gamma j} - \bar{Z}_{\delta j} + ([Z_{\gamma\alpha}] - [Z_{\delta\alpha}] - [Z_{\gamma\beta}] + [Z_{\delta\beta}])\overline{\Delta i}_{\alpha\beta}\} \qquad (4.3.7)$$

Solving equation (4.3.7) for $\overline{\Delta i}_{\alpha\beta}$,

$$\overline{\Delta i}_{\alpha\beta} = \{U - ([y_s] - [y'_s])([Z_{\gamma\alpha}] - [Z_{\delta\alpha}] - [Z_{\gamma\beta}] + [Z_{\delta\beta}])\}^{-1} ([y_s] - [y'_s])(\bar{Z}_{\gamma j} - \bar{Z}_{\delta j}) \qquad (4.3.8)$$

Designating

$$[\Delta y_s] = [y_s] - [y'_s]$$

and

$$[M] = \{U - [\Delta y_s]([Z_{\gamma\alpha}] - [Z_{\delta\alpha}] - [Z_{\gamma\beta}] + [Z_{\delta\beta}])\}$$

equation (4.3.8) becomes

$$\overline{\Delta i}_{\alpha\beta} = [M]^{-1}[\Delta y_s](\bar{Z}_{\gamma j} - \bar{Z}_{\delta j}) \qquad (4.3.9)$$

Substituting from equation (4.3.9) for $\overline{\Delta i}_{\alpha\beta}$ into (4.3.4),

$$E'_i = Z_{ij} + (\bar{Z}_{i\alpha} - \bar{Z}_{i\beta})[M]^{-1}[\Delta y_s](\bar{Z}_{\gamma j} - \bar{Z}_{\delta j})$$

This equation gives, for the original network, the voltage at bus i as a result of injecting one per unit current at bus j and the appropriate current changes at buses μ, ν, ρ, and σ to simulate the effect of changes in the elements μ–ν. Thus, from the definition of the bus impedance matrix, the ijth element of the modified bus impedance matrix is

$$Z'_{ij} = Z_{ij} + (\bar{Z}_{i\alpha} - \bar{Z}_{i\beta})[M]^{-1}[\Delta y_s](\bar{Z}_{\gamma j} - \bar{Z}_{\delta j}) \qquad i = 1, 2, \ldots, n$$

The process is repeated for each $j = 1, 2, \ldots, n$ to obtain all elements of Z'_{BUS}.

4.4 Example of formation and modification of bus impedance matrix

The method based on the algorithm for forming the bus impedance matrix will be illustrated using the sample network given in Fig. 3.10.

Examples of the modification of this bus impedance matrix will also be given.

Problem

a. Form the bus impedance matrix Z_{BUS} of the network shown in Fig. 4.5.

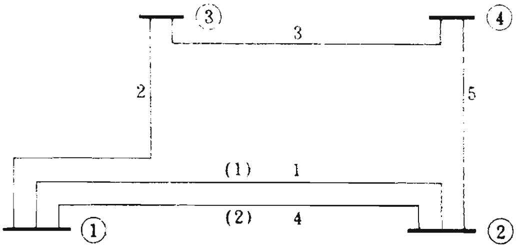

Fig. 4.5 Sample network.

b. Modify the bus impedance matrix obtained in part *a* to include the addition of an element from bus 2 to bus 4 with an impedance of 0.3 and coupled to element 5 with a mutual impedance of 0.1.

c. Modify the bus impedance matrix obtained in part *b* to remove the new element from bus 2 to bus 4.

Solution

The data for the network is given in Table 4.2. The bus impedance matrix will be formed by adding elements of the network in the order indicated in the first column of this table. Node 1 is selected as reference.

Table 4.2 Impedances for sample network

	Self		*Mutual*	
Element number	*Bus code* $p-q$	*Impedance* $z_{pq,pq}$	*Bus code* $r-s$	*Impedance* $z_{pq,rs}$
1	1-2(1)	0.6		
4	1-2(2)	0.4	1-2(1)	0.2
2	1-3	0.5	1-2(1)	0.1
3	3-4	0.5		
5	2-4	0.2		

a. Step 1. Start with element 1 which is a branch from $p = 1$ to $q = 2$. The elements of the bus impedance matrix for the partial network containing the single branch are

$$Z_{BUS} = \begin{array}{c|c|c|} & \textcircled{1} & \textcircled{2} \\ \hline \textcircled{1} & 0 & 0 \\ \hline \textcircled{2} & 0 & 0.6 \\ \hline \end{array}$$

Since node 1 is the reference, the elements of the first row and column are zero and need not be written. Thus

$$Z_{BUS} = \begin{array}{c|c|} & \textcircled{2} \\ \hline \textcircled{2} & 0.6 \\ \hline \end{array}$$

Step 2. Add element 4, which is a link, from $p = 1$ (reference) to $q = 2$, mutually coupled with element 1. The augmented impedance matrix with the fictitious node l will be

$$\begin{array}{c|c|c|} & \textcircled{2} & l \\ \hline \textcircled{2} & 0.6 & Z_{2l} \\ \hline l & Z_{l2} & Z_{ll} \\ \hline \end{array}$$

where

$$Z_{2l} = Z_{l2} = -Z_{22} + \frac{y_{12(2),12(1)}(Z_{12} - Z_{22})}{y_{12(2),12(2)}}$$

$$Z_{ll} = -Z_{2l} + \frac{1 + y_{12(2),12(1)}(Z_{1l} - Z_{2l})}{y_{12(2),12(2)}}$$

and $Z_{12} = Z_{1l} = 0$. Invert the primitive impedance matrix of the partial network to obtain the primitive admittance matrix.

$$[z_{\rho\sigma,\rho\sigma}] = \begin{array}{c|c|c|} & 1\text{–}2(1) & 1\text{–}2(2) \\ \hline 1\text{–}2(1) & 0.6 & 0.2 \\ \hline 1\text{–}2(2) & 0.2 & 0.4 \\ \hline \end{array}$$

$$[z_{\rho\sigma,\rho\sigma}]^{-1} = [y_{\rho\sigma,\rho\sigma}] = \begin{array}{c|c|c|} & 1\text{–}2(1) & 1\text{–}2(2) \\ \hline 1\text{–}2(1) & 2 & -1 \\ \hline 1\text{–}2(2) & -1 & 3 \\ \hline \end{array}$$

Then,

$$Z_{2l} = Z_{l2} = -0.6 + \frac{(-1)(-0.6)}{3} = -0.4$$

$$Z_{ll} = 0.4 + \frac{1-1(0.4)}{3} = 0.6$$

and the augmented matrix is

$$\begin{array}{c|c|c|} & ② & l \\ \hline ② & 0.6 & -0.4 \\ \hline l & -0.4 & 0.6 \\ \hline \end{array}$$

Eliminating the lth row and column,

$$Z'_{22} = Z_{22} - \frac{Z_{2l}Z_{l2}}{Z_{ll}} = 0.6 - \frac{(-0.4)(-0.4)}{0.6} = 0.3333$$

and

$$Z_{BUS} = \begin{array}{c|c|} & ② \\ \hline ② & 0.3333 \\ \hline \end{array}$$

Step 3. Add element 2, which is a branch, from $p = 1$ (reference) to $q = 3$, mutually coupled with element 1. This adds a new bus and the bus impedance matrix is

$$Z_{BUS} = \begin{array}{c|c|c|} & ② & ③ \\ \hline ② & 0.3333 & Z_{23} \\ \hline ③ & Z_{32} & Z_{33} \\ \hline \end{array}$$

where

$$Z_{32} = Z_{23} = \frac{\begin{bmatrix} y_{13,12(1)} & y_{13,12(2)} \end{bmatrix} \begin{bmatrix} Z_{12} - Z_{22} \\ Z_{12} - Z_{22} \end{bmatrix}}{y_{13,13}}$$

$$Z_{33} = \frac{1 + \begin{bmatrix} y_{13,12(1)} & y_{13,12(2)} \end{bmatrix} \begin{bmatrix} Z_{13} - Z_{23} \\ Z_{13} - Z_{23} \end{bmatrix}}{y_{13,13}}$$

and $Z_{12} = Z_{13} = 0$. Invert the primitive impedance matrix to obtain the primitive admittance matrix.

$$
[z_{p\sigma,\rho\sigma}] = \begin{array}{c|c|c|c|}
 & \text{1–2(1)} & \text{1–2(2)} & \text{1–3} \\
\hline
\text{1–2(1)} & 0.6 & 0.2 & 0.1 \\
\hline
\text{1–2(2)} & 0.2 & 0.4 & \\
\hline
\text{1–3} & 0.1 & & 0.5 \\
\hline
\end{array}
$$

$$
[z_{\rho\sigma,\rho\sigma}]^{-1} = [y_{\rho\sigma,\rho\sigma}] = \begin{array}{c|c|c|c|}
 & \text{1–2(1)} & \text{1–2(2)} & \text{1–3} \\
\hline
\text{1–2(1)} & 2.0833 & -1.0417 & -0.4167 \\
\hline
\text{1–2(2)} & -1.0417 & 3.0208 & 0.2083 \\
\hline
\text{1–3} & -0.4167 & 0.2083 & 2.0833 \\
\hline
\end{array}
$$

Then,

$$
Z_{32} = Z_{23} = \frac{\begin{bmatrix} -0.4167 & 0.2083 \end{bmatrix} \begin{bmatrix} -0.3333 \\ -0.3333 \end{bmatrix}}{2.0833} = 0.0333
$$

$$
Z_{33} = \frac{1 + \begin{bmatrix} -0.4167 & 0.2083 \end{bmatrix} \begin{bmatrix} -0.0333 \\ -0.0333 \end{bmatrix}}{2.0833} = 0.4833
$$

and

$$
Z_{BUS} = \begin{array}{c|c|c|}
 & ② & ③ \\
\hline
② & 0.3333 & 0.0333 \\
\hline
③ & 0.0333 & 0.4833 \\
\hline
\end{array}
$$

Step 4. Add element 3, which is a branch, from $p = 3$ to $q = 4$, not mutually coupled. This adds a new bus.

$Z_{24} = Z_{42} = Z_{32} = 0.0333$

$Z_{34} = Z_{43} = Z_{33} = 0.4833$

$Z_{44} = Z_{34} + z_{34,34} = 0.4833 + 0.5 = 0.9833$

Thus,

$$Z_{BUS} = \begin{array}{c|c|c|c} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline \textcircled{2} & 0.3333 & 0.0333 & 0.0333 \\ \hline \textcircled{3} & 0.0333 & 0.4833 & 0.4833 \\ \hline \textcircled{4} & 0.0333 & 0.4833 & 0.9833 \end{array}$$

Step 5. Add element 5, which is a link, from $p = 2$ to $q = 4$, not mutually coupled. The elements of the lth row and column of the augmented matrix are

$$Z_{2l} = Z_{l2} = Z_{22} - Z_{42} = 0.3333 - 0.0333 = 0.3000$$
$$Z_{3l} = Z_{l3} = Z_{23} - Z_{43} = 0.0333 - 0.4833 = -0.4500$$
$$Z_{4l} = Z_{l4} = Z_{24} - Z_{44} = 0.0333 - 0.9833 = -0.9500$$
$$Z_{ll} = Z_{2l} - Z_{4l} + z_{24,24} = 0.3000 + 0.9500 + 0.2 = 1.4500$$

The augmented matrix is

$$\begin{array}{c|c|c|c|c} & \textcircled{2} & \textcircled{3} & \textcircled{4} & l \\ \hline \textcircled{2} & 0.3333 & 0.0333 & 0.0333 & 0.3000 \\ \hline \textcircled{3} & 0.0333 & 0.4833 & 0.4833 & -0.4500 \\ \hline \textcircled{4} & 0.0333 & 0.4833 & 0.9833 & -0.9500 \\ \hline l & 0.3000 & -0.4500 & -0.9500 & 1.4500 \end{array}$$

Eliminating the lth row and column,

$$Z'_{22} = Z_{22} - \frac{Z_{2l}Z_{l2}}{Z_{ll}} = 0.3333 - \frac{(0.3000)(0.3000)}{1.4500} = 0.2712$$

$$Z'_{23} = Z'_{32} = Z_{23} - \frac{Z_{2l}Z_{l3}}{Z_{ll}} = 0.0333 - \frac{(0.3000)(-0.4500)}{1.4500} = 0.1263$$

$$Z'_{24} = Z'_{42} = Z_{24} - \frac{Z_{2l}Z_{l4}}{Z_{ll}} = 0.0333 - \frac{(0.3000)(-0.9500)}{1.4500} = 0.2298$$

$$Z'_{33} = Z_{33} - \frac{Z_{3l}Z_{l3}}{Z_{ll}} = 0.4833 - \frac{(-0.4500)(-0.4500)}{1.4500} = 0.3436$$

$$Z'_{34} = Z'_{43} = Z_{34} - \frac{Z_{3l}Z_{l4}}{Z_{ll}} = 0.4833 - \frac{(-0.4500)(-0.9500)}{1.4500} = 0.1885$$

$$Z'_{44} = Z_{44} - \frac{Z_{4l}Z_{l4}}{Z_{ll}} = 0.9833 - \frac{(-0.9500)(-0.9500)}{1.4500} = 0.3609$$

and

$$Z_{BUS} = \begin{array}{c|c|c|c|} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline \textcircled{2} & 0.2712 & 0.1263 & 0.2298 \\ \hline \textcircled{3} & 0.1263 & 0.3436 & 0.1885 \\ \hline \textcircled{4} & 0.2298 & 0.1885 & 0.3609 \\ \hline \end{array}$$

b. Adding a new element, which is a link, from $p = 2$ to $q = 4$, mutually coupled with element 5 results in the augmented matrix

$$\begin{array}{c|c|c|c|c|} & \textcircled{2} & \textcircled{3} & \textcircled{4} & l \\ \hline \textcircled{2} & 0.2712 & 0.1263 & 0.2298 & Z_{2l} \\ \hline \textcircled{3} & 0.1263 & 0.3436 & 0.1885 & Z_{3l} \\ \hline \textcircled{4} & 0.2298 & 0.1885 & 0.3609 & Z_{4l} \\ \hline l & Z_{l2} & Z_{l3} & Z_{l4} & Z_{ll} \\ \hline \end{array}$$

where

$$Z_{li} = Z_{pi} - Z_{qi} + \frac{\bar{y}_{pq,\rho\sigma}(\bar{Z}_{\rho i} - \bar{Z}_{\sigma i})}{y_{pq,pq}} \qquad i = 2, 3, 4$$

and

$$Z_{ll} = Z_{pl} - Z_{ql} + \frac{1 + \bar{y}_{pq,\rho\sigma}(\bar{Z}_{\rho l} - \bar{Z}_{\sigma l})}{y_{pq,pq}}$$

The primitive impedance matrix is

$$[z] = \begin{array}{c|c|c|c|c|c|c|} & 1\text{–}2(1) & 1\text{–}2(2) & 1\text{–}3 & 3\text{–}4 & 2\text{–}4(1) & 2\text{–}4(2) \\ \hline 1\text{–}2(1) & 0.6 & 0.2 & 0.1 & & & \\ \hline 1\text{–}2(2) & 0.2 & 0.4 & & & & \\ \hline 1\text{–}3 & 0.1 & & 0.5 & & & \\ \hline 3\text{–}4 & & & & 0.5 & & \\ \hline 2\text{–}4(1) & & & & & 0.2 & 0.1 \\ \hline 2\text{–}4(2) & & & & & 0.1 & 0.3 \\ \hline \end{array}$$

Since the new element is coupled to only one other element, it is sufficient to invert the submatrix for the coupled elements, which is

$$[z_{p\sigma,\rho\tau}] = \begin{array}{c|c|c|} & \text{2-4(1)} & \text{2-4(2)} \\ \hline \text{2-4(1)} & 0.2 & 0.1 \\ \hline \text{2-4(2)} & 0.1 & 0.3 \\ \hline \end{array}$$

Thus

$$[y_{p\sigma,p\sigma}] = \begin{array}{c|c|c|} & \text{2-4(1)} & \text{2-4(2)} \\ \hline \text{2-4(1)} & 6 & -2 \\ \hline \text{2-4(2)} & -2 & 4 \\ \hline \end{array}$$

and

$$Z_{2l} = Z_{l2} = 0.2712 - 0.2298 + \frac{-2.0(0.2712 - 0.2298)}{4.0} = 0.0207$$

$$Z_{3l} = Z_{l3} = 0.1263 - 0.1885 + \frac{-2.0(0.1263 - 0.1885)}{4.0} = -0.0311$$

$$Z_{4l} = Z_{l4} = 0.2298 - 0.3609 + \frac{-2.0(0.2298 - 0.3609)}{4.0} = -0.0656$$

$$Z_{ll} = 0.0207 + 0.0656 + \frac{1 - 2.0(0.0207 + 0.0656)}{4.0} = 0.2931$$

and the augmented matrix is

$$\begin{array}{c|c|c|c|c|} & \textcircled{2} & \textcircled{3} & \textcircled{4} & l \\ \hline \textcircled{2} & 0.2712 & 0.1263 & 0.2298 & 0.0207 \\ \hline \textcircled{3} & 0.1263 & 0.3436 & 0.1885 & -0.0311 \\ \hline \textcircled{4} & 0.2298 & 0.1885 & 0.3609 & -0.0656 \\ \hline l & 0.0207 & -0.0311 & -0.0656 & 0.2931 \\ \hline \end{array}$$

Eliminating the lth row and column,

$$Z'_{22} = 0.2712 - \frac{(0.0207)(0.0207)}{0.2931} = 0.2697$$

$$Z'_{23} = Z'_{32} = 0.1263 - \frac{(0.0207)(-0.0311)}{0.2931} = 0.1285$$

$$Z'_{24} = Z'_{42} = 0.2298 - \frac{(0.0207)(-0.0656)}{0.2931} = 0.2344$$

$$Z'_{33} = 0.3436 - \frac{(-0.0311)(-0.0311)}{0.2931} = 0.3403$$

$$Z'_{34} = Z'_{43} = 0.1885 - \frac{(-0.0311)(-0.0656)}{0.2931} = 0.1816$$

$$Z'_{44} = 0.3609 - \frac{(-0.0656)(-0.0656)}{0.2931} = 0.3462$$

Finally,

$$Z_{BUS} = \begin{array}{c|c|c|c|} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline \textcircled{2} & 0.2697 & 0.1285 & 0.2344 \\ \hline \textcircled{3} & 0.1285 & 0.3403 & 0.1816 \\ \hline \textcircled{4} & 0.2344 & 0.1816 & 0.3462 \\ \hline \end{array}$$

c. The modified elements of this bus impedance matrix for the removal of the network element 2–4(2) mutually coupled to network element 2–4(1) are obtained from

$$Z'_{ij} = Z_{ij} + (\bar{Z}_{i\alpha} - \bar{Z}_{i\beta})[M]^{-1}[\Delta y_s](\bar{Z}_{\gamma j} - \bar{Z}_{\delta j}) \qquad i, j = 2, 3, 4$$

where μ–ν is 2–4 and ρ–σ is also 2–4 and the indices α, γ = 2, 2 and β, δ = 4, 4.

The original primitive admittance submatrix is

$$[y_s] = \begin{array}{c|c|c|} & \text{2–4(1)} & \text{2–4(2)} \\ \hline \text{2–4(1)} & 6 & -2 \\ \hline \text{2–4(2)} & -2 & 4 \\ \hline \end{array}$$

and the modified primitive admittance submatrix is

$$[y'_s] = \begin{array}{c|c|c|} & \text{2–4(1)} & \text{2–4(2)} \\ \hline \text{2–4(1)} & 5 & \\ \hline \text{2–4(2)} & & \\ \hline \end{array}$$

Thus

$$[y_s] - [y'_s] = [\Delta y_s] = \begin{array}{c|c|c|} & \text{2–4(1)} & \text{2–4(2)} \\ \hline \text{2–4(1)} & 1 & -2 \\ \hline \text{2–4(2)} & -2 & 4 \\ \hline \end{array}$$

Also

$$[M] = [U - [\Delta y_s]([Z_{\gamma\alpha}] - [Z_{\delta\alpha}] - [Z_{\gamma\beta}] + [Z_{\delta\beta}])]$$

where

$$[Z_{\gamma\alpha}] = \begin{array}{c|c|c|} & \text{②} & \text{②} \\ \hline \text{②} & 0.2697 & 0.2697 \\ \hline \text{②} & 0.2697 & 0.2697 \\ \hline \end{array}$$

$$[Z_{\delta\alpha}] = \begin{array}{c|c|c|} & \text{②} & \text{②} \\ \hline \text{④} & 0.2344 & 0.2344 \\ \hline \text{④} & 0.2344 & 0.2344 \\ \hline \end{array}$$

$$[Z_{\gamma\beta}] = \begin{array}{c|c|c|} & \text{④} & \text{④} \\ \hline \text{②} & 0.2344 & 0.2344 \\ \hline \text{②} & 0.2344 & 0.2344 \\ \hline \end{array}$$

$$[Z_{\delta\beta}] = \begin{array}{c|c|c|} & \text{④} & \text{④} \\ \hline \text{④} & 0.3462 & 0.3462 \\ \hline \text{④} & 0.3462 & 0.3462 \\ \hline \end{array}$$

Substituting in the above equation,

$$[M] = \begin{bmatrix} 1.1471 & 0.1471 \\ -0.2942 & 0.7058 \end{bmatrix}$$

$$[M]^{-1} = \begin{bmatrix} 0.82753 & -0.17247 \\ 0.34494 & 1.34494 \end{bmatrix}$$

and

$$[M]^{-1}[\Delta y_s] = \begin{bmatrix} 1.17247 & -2.34494 \\ -2.34494 & 4.68988 \end{bmatrix}$$

For $i = 2$ and $j = 2$,

$$Z'_{22} = Z_{22} + \left(\begin{bmatrix} Z_{22} & Z_{22} \end{bmatrix} - \begin{bmatrix} Z_{24} & Z_{24} \end{bmatrix} \right) [M]^{-1}[\Delta y_s] \left(\begin{bmatrix} Z_{22} \\ Z_{22} \end{bmatrix} - \begin{bmatrix} Z_{42} \\ Z_{42} \end{bmatrix} \right)$$

$$Z'_{22} = 0.2697 + \begin{bmatrix} 0.0353 & 0.0353 \end{bmatrix} \begin{bmatrix} 1.17247 & -2.34494 \\ -2.34494 & 4.68988 \end{bmatrix} \begin{bmatrix} 0.0353 \\ 0.0353 \end{bmatrix}$$

$$Z'_{22} = 0.2697 + 0.0015 = 0.2712$$

For $i = 2$ and $j = 3$,

$$Z'_{23} = Z_{23} + \left(\begin{bmatrix} Z_{22} & Z_{22} \end{bmatrix} - \begin{bmatrix} Z_{24} & Z_{24} \end{bmatrix} \right) [M]^{-1}[\Delta y_s] \left(\begin{bmatrix} Z_{23} \\ Z_{23} \end{bmatrix} - \begin{bmatrix} Z_{43} \\ Z_{43} \end{bmatrix} \right)$$

$$Z'_{23} = 0.1285 + \begin{bmatrix} 0.0353 & 0.0353 \end{bmatrix} \begin{bmatrix} 1.17247 & -2.34494 \\ -2.34494 & 4.68988 \end{bmatrix} \begin{bmatrix} -0.0531 \\ -0.0531 \end{bmatrix}$$

$$Z'_{23} = 0.1285 - 0.0022 = 0.1263$$

For $i = 2$ and $j = 4$,

$$Z'_{24} = Z_{24} + \left(\begin{bmatrix} Z_{22} & Z_{22} \end{bmatrix} - \begin{bmatrix} Z_{24} & Z_{24} \end{bmatrix} \right) [M]^{-1}[\Delta y_s] \left(\begin{bmatrix} Z_{24} \\ Z_{24} \end{bmatrix} - \begin{bmatrix} Z_{44} \\ Z_{44} \end{bmatrix} \right)$$

$$Z'_{24} = 0.2344 + \begin{bmatrix} 0.0353 & 0.0353 \end{bmatrix} \begin{bmatrix} 1.17247 & -2.34494 \\ -2.34494 & 4.68988 \end{bmatrix} \begin{bmatrix} -0.1118 \\ -0.1118 \end{bmatrix}$$

$$Z'_{24} = 0.2344 - 0.0046 = 0.2298$$

For $i = 3$ and $j = 3$,

$$Z'_{33} = Z_{33} + \left(\begin{bmatrix} Z_{32} & Z_{32} \end{bmatrix} - \begin{bmatrix} Z_{34} & Z_{34} \end{bmatrix} \right) [M]^{-1}[\Delta y_s] \left(\begin{bmatrix} Z_{23} \\ Z_{23} \end{bmatrix} - \begin{bmatrix} Z_{43} \\ Z_{43} \end{bmatrix} \right)$$

$$Z'_{33} = 0.3403 + \begin{bmatrix} -0.0531 & -0.0531 \end{bmatrix} \begin{bmatrix} 1.17247 & -2.34494 \\ -2.34494 & 4.68988 \end{bmatrix} \begin{bmatrix} -0.0531 \\ -0.0531 \end{bmatrix}$$

$$Z'_{33} = 0.3403 + 0.0033 = 0.3436$$

For $i = 3$ and $j = 4$,

$$Z'_{34} = Z_{34} + \left(\begin{bmatrix} Z_{32} & Z_{32} \end{bmatrix} - \begin{bmatrix} Z_{34} & Z_{34} \end{bmatrix} \right) [M]^{-1}[\Delta y_s] \left(\begin{bmatrix} Z_{24} \\ Z_{24} \end{bmatrix} - \begin{bmatrix} Z_{44} \\ Z_{44} \end{bmatrix} \right)$$

$$Z'_{34} = 0.1816 + \begin{bmatrix} -0.0531 & -0.0531 \end{bmatrix} \begin{bmatrix} 1.17247 & -2.34494 \\ -2.34494 & 4.68988 \end{bmatrix} \begin{bmatrix} -0.1118 \\ -0.1118 \end{bmatrix}$$

$$Z'_{34} = 0.1816 + 0.0069 = 0.1885$$

For $i = 4$ and $j = 4$,

$$Z'_{44} = Z_{44} + \left(\begin{array}{c|c} Z_{42} & Z_{42} \end{array} - \begin{array}{c|c} Z_{44} & Z_{44} \end{array} \right) [M]^{-1}[\Delta y_s] \left(\begin{array}{c} Z_{24} \\ \hline Z_{24} \end{array} - \begin{array}{c} Z_{44} \\ \hline Z_{44} \end{array} \right)$$

$$Z'_{44} = 0.3462 + \begin{bmatrix} -0.1118 & -0.1118 \end{bmatrix} \begin{bmatrix} 1.17247 & -2.34494 \\ -2.34494 & 4.68988 \end{bmatrix} \begin{bmatrix} -0.1118 \\ -0.1118 \end{bmatrix}$$

$$Z'_{44} = 0.3462 + 0.0147 = 0.3609$$

Thus

$$Z_{BUS} = \begin{array}{c|c|c|c} & ② & ③ & ④ \\ \hline ② & 0.2712 & 0.1263 & 0.2298 \\ \hline ③ & 0.1263 & 0.3436 & 0.1885 \\ \hline ④ & 0.2298 & 0.1885 & 0.3609 \end{array}$$

which is equal to the matrix obtained in part *a*.

4.5 *Derivation of loop admittance matrix from bus impedance matrix*

Derivation of node-pair impedance matrix from bus impedance matrix

An element of the node-pair impedance matrix Z_{NP} is designated by $Z_{ij,pq}$. If there is a current source between p and q only, an element of this matrix is defined as

$$Z_{ij,pq} = \frac{E_i - E_j}{I_{pq}} \qquad \begin{array}{c} i, j = 1, 2, \ldots, n \\ i \neq j \end{array} \tag{4.5.1}$$

Letting the current $I_{pq} = 1$ per unit as shown in Fig. 4.6,

$$Z_{ij,pq} = E_i - E_j$$

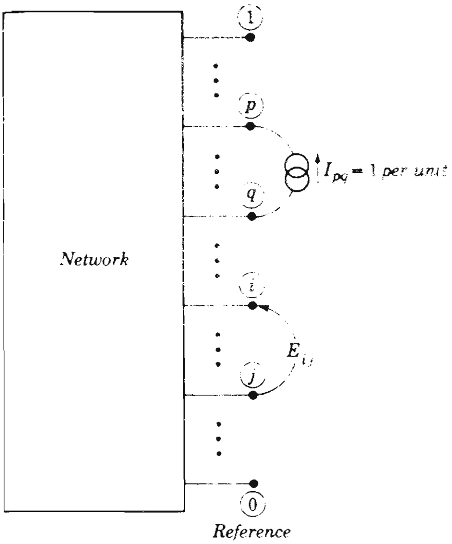

Fig. 4.6 Injected current for calculation of $Z_{ij,pq}$.

The performance equation of the network written in terms of Z_{BUS} is

$$
\begin{array}{c} \\ \hline E_1 \\ \cdots \\ E_p \\ \cdots \\ E_q \\ \cdots \\ E_n \\ \hline \end{array}
=
\begin{array}{c|ccccccc|}
 & 1 & & p & & q & & n \\ \hline
1 & Z_{11} & \cdots & Z_{1p} & \cdots & Z_{1q} & \cdots & Z_{1n} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
p & Z_{p1} & \cdots & Z_{pp} & \cdots & Z_{pq} & \cdots & Z_{pn} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
q & Z_{q1} & \cdots & Z_{qp} & \cdots & Z_{qq} & \cdots & Z_{qn} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
n & Z_{n1} & \cdots & Z_{np} & \cdots & Z_{nq} & \cdots & Z_{nn} \\ \hline
\end{array}
\begin{array}{c} \\ \hline I_1 \\ \cdots \\ I_p \\ \cdots \\ I_q \\ \cdots \\ I_n \\ \hline \end{array}
\qquad (4.5.2)
$$

Since $I_{pq} = 1$, then $I_p = 1$ and $I_q = -1$. From equation (4.5.2) it follows that

$$E_i = Z_{ip} - Z_{iq} \qquad \text{and} \qquad E_j = Z_{jp} - Z_{jq}$$

From equation (4.5.1),

$$Z_{ij,pq} = Z_{ip} - Z_{jp} - Z_{iq} + Z_{jq} \qquad \begin{matrix} i, j = 1, 2, \ldots n \\ i \neq j \end{matrix} \tag{4.5.3}$$

Using all node-pair combinations for p–q, all elements of Z_{NP} can be calculated from equation (4.5.3). The matrix Z_{NP} has dimension

$$\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$$

Derivation of element-pair admittance matrix from node-pair impedance matrix

The element-pair admittance matrix is designated by Y_{EP} and the element of the matrix by $Y_{ij,pq}$. If there is a voltage source only in series with p–q, an element of this matrix is defined as

$$Y_{ij,pq} = \frac{i_{ij}}{e_{pq}}$$

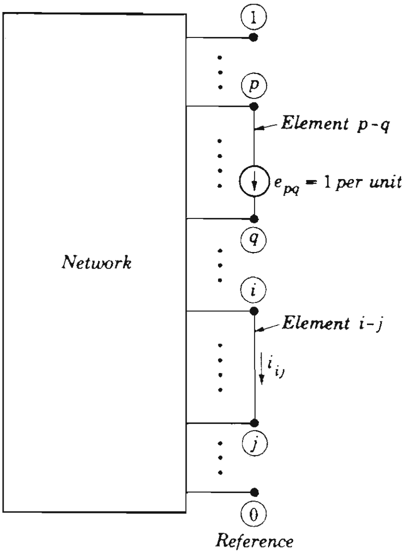

***Fig. 4.7** Series voltage source for calculation of $Y_{ij,pq}$.*

where i_{ij} = current through the element i–j

e_{pq} = voltage source in series with the element p–q

Let e_{pq} = 1 per unit, as shown in Fig. 4.7, then

$$Y_{ij,pq} = i_{ij} \tag{4.5.4}$$

It remains therefore to calculate the current i_{ij}.

The performance equation in admittance form for the primitive network is

$$\bar{i} + \bar{j} = [y]\bar{v}$$

The current through the element i–j is

$$i_{ij} = -j_{ij} + \bar{y}_{ij,\rho\sigma}\bar{v}_{\rho\sigma} \tag{4.5.5}$$

where $\rho\sigma$ refers to all elements of the network. The voltage source in series with p–q induces currents in the elements mutually coupled with p–q. This voltage source can be replaced by equivalent current sources in parallel with each element, as shown in Fig. 4.8. The equivalent

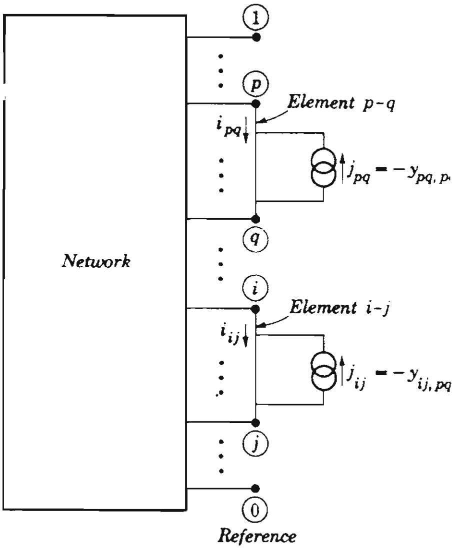

***Fig. 4.8** Equivalent source currents for calculation of $Y_{ij,pq}$.*

current source for the element i–j is

$$j_{ij} = -y_{ij,pq} \tag{4.5.6}$$

The voltages $\bar{v}_{\rho\sigma}$ can be obtained from the performance equation of the network using the node-pair impedance matrix

$$\bar{E}_{NP} = Z_{NP}\bar{I}_{NP} \tag{4.5.7}$$

From equation (4.5.7), the voltages across the elements ρ–σ of the network are

$$\bar{E}_\rho - \bar{E}_\sigma = [Z_{\rho\sigma,\mu\nu}]\bar{I}_{\mu\nu} \tag{4.5.8}$$

where

$$\bar{v}_{\rho\sigma} = \bar{E}_\rho - \bar{E}_\sigma \tag{4.5.9}$$

and the indices $\rho\sigma$ and $\mu\nu$ refer to the node-pairs corresponding to the terminals of the network elements. The elements of $Z_{\rho\sigma,\mu\nu}$ are obtained directly from Z_{NP}, and $Z_{\rho\sigma,\mu\nu}$ has dimension $e \times e$ where e equals the number of elements. The elements of the vector $\bar{I}_{\mu\nu}$ are equal to the shunt source currents which replace the series source voltage. Therefore,

$$\bar{I}_{\mu\nu} = -\bar{y}_{\mu\nu,pq} \tag{4.5.10}$$

Substituting from equations (4.5.9) and (4.5.10) into (4.5.8),

$$\bar{v}_{\rho\sigma} = -[Z_{\rho\sigma,\mu\nu}]\bar{y}_{\mu\nu,pq} \tag{4.5.11}$$

Substituting from equations (4.5.6) and (4.5.11) into (4.5.5) yields

$$i_{ij} = y_{ij,pq} - \bar{y}_{ij,\rho\sigma}[Z_{\rho\sigma,\mu\nu}]\bar{y}_{\mu\nu,pq}$$

Hence from equation (4.5.4),

$$Y_{ij,pq} = y_{ij,pq} - \bar{y}_{ij,\rho\sigma}[Z_{\rho\sigma,\mu\nu}]\bar{y}_{\mu\nu,pq} \tag{4.5.12}$$

and using all combinations of element pairs the matrix Y_{EP} can be obtained. The matrix Y_{EP} has dimension $e \times e$.

If there is no mutual coupling in the network, equation (4.5.12) reduces to

$$Y_{ij,pq} = -y_{ij,ij}Z_{ij,pq}y_{pq,pq} \qquad ij \neq pq$$

and

$$Y_{ij,ij} = y_{ij,ij} - y_{ij,ij} Z_{ij,ij} y_{ij,ij}$$

Equation (4.5.12) can be written in terms of the elements of the bus impedance matrix since

$$[Z_{\rho\sigma,\mu\nu}] = [Z_{\rho\mu}] - [Z_{\sigma\mu}] - [Z_{\rho\nu}] + [Z_{\sigma\nu}]$$

Then

$$Y_{ij,pq} = y_{ij,pq} - \bar{y}_{ij,\rho\sigma}([Z_{\rho\mu}] - [Z_{\sigma\mu}] - [Z_{\rho\nu}] + [Z_{\sigma\nu}])\bar{y}_{\mu\nu,pq}$$

If it is desired to derive Z_{NP} from a given Y_{EP}, the elements of Z_{NP} can be obtained in a manner similar to that described in this section. An element of Z_{NP}, in terms of the element-pair admittance matrix, can be expressed by

$$Z_{ij,pq} = z_{ij,pq} - \bar{z}_{ij,\rho\sigma}[Y_{\rho\sigma,\mu\nu}]\bar{z}_{\mu\nu,pq} \tag{4.5.13}$$

If there is no mutual coupling, equation (4.5.13) reduces to

$$Z_{ij,pq} = -z_{ij,ij} Y_{ij,pq} z_{pq,pq} \qquad ij \neq pq$$

and

$$Z_{ij,ij} = z_{ij,ij} - z_{ij,ij} Y_{ij,ij} z_{ij,ij}$$

Derivation of loop admittance matrix from element-pair admittance matrix

Given the element-pair admittance matrix Y_{EP}, the elements of the loop admittance matrix Y_{LOOP} can be obtained directly. Y_{EP} is partitioned as follows:

$$Y_{EP} = \begin{array}{c|c|c} & \text{Branches} & \text{Links} \\ \hline \text{Branches} & Y_1 & Y_2 \\ \hline \text{Links} & Y_3 & Y_4 \end{array}$$

where the submatrix Y_4 is associated with the basic loops of the interconnected network since each link corresponds to a basic loop. Therefore

$$Y_{LOOP} = Y_4$$

It is possible to derive Y_{EP} from a given Y_{LOOP} since the elements of the submatrices Y_1, Y_2, and Y_3 can be determined from the elements of Y_4. Let i–j be a branch common to loops A, B, and C and let p–q be a branch common to loops G and H. If a voltage source $e_{pq} = 1$ per unit is applied in series with the branch p–q, then by definition of an element of Y_{EP},

$$Y_{ij,pq} = i_{ij}$$

Moreover, the voltages in the loops G and H are equal to one per unit. Hence, the currents in the loops A, B, and C are

$$I_A = Y_{AG} + Y_{AH}$$
$$I_B = Y_{BG} + Y_{BH}$$
$$I_C = Y_{CG} + Y_{CH}$$

where the admittances are obtained from the loop admittance matrix. Since the current in branch i–j is the algebraic sum of the currents in loops A, B, and C,

$$i_{ij} = I_A + I_B + I_C$$

Therefore,

$$Y_{ij,pq} = Y_{AG} + Y_{AH} + Y_{BG} + Y_{BH} + Y_{CG} + Y_{CH}$$

The signs of the loop admittance terms are determined by the orientation of the branches with respect to the loops.

4.6 Example of derivation of loop admittance matrix from bus impedance matrix

The method of deriving the loop admittance matrix from the bus impedance matrix will be illustrated for the sample network shown in Fig. 3.10.

Problem

Derive the loop admittance matrix Y_{LOOP} from the bus impedance matrix Z_{BUS} of the network shown in Fig. 3.10.

Solution

The primitive admittance matrix is

$$[y] = \begin{array}{c|c|c|c|c|c|} & \text{1-2(1)} & \text{1-2(2)} & \text{1-3} & \text{2-4} & \text{3-4} \\ \hline \text{1-2(1)} & 2.083 & -1.042 & -0.417 & & \\ \hline \text{1-2(2)} & -1.042 & 3.021 & 0.208 & & \\ \hline \text{1-3} & -0.417 & 0.208 & 2.083 & & \\ \hline \text{2-4} & & & & 5.0 & \\ \hline \text{3-4} & & & & & 2.0 \\ \hline \end{array}$$

The bus impedance matrix of the network obtained by nonsingular transformation is

$$Z_{BUS} = \begin{array}{c|c|c|c|} & ② & ③ & ④ \\ \hline ② & 0.271 & 0.126 & 0.230 \\ \hline ③ & 0.126 & 0.344 & 0.189 \\ \hline ④ & 0.230 & 0.189 & 0.362 \\ \hline \end{array}$$

First, form the node-pair impedance matrix Z_{NP}. The elements of the first row from the equation

$$Z_{ij,pq} = Z_{ip} - Z_{jp} - Z_{iq} + Z_{jq}$$

are

$$\begin{aligned} Z_{12,12} &= Z_{11} - Z_{21} - Z_{12} + Z_{22} \\ &= 0 - 0 - 0 + 0.271 = 0.271 \\ Z_{12,13} &= Z_{11} - Z_{21} - Z_{13} + Z_{23} \\ &= 0 - 0 - 0 + 0.126 = 0.126 \\ Z_{12,24} &= Z_{12} - Z_{22} - Z_{14} + Z_{24} \\ &= 0 - 0.271 - 0 + 0.230 = -0.041 \\ Z_{12,34} &= Z_{13} - Z_{23} - Z_{14} + Z_{24} \\ &= 0 - 0.126 - 0 + 0.230 = 0.104 \\ Z_{12,14} &= Z_{11} - Z_{21} - Z_{14} + Z_{24} \\ &= 0 - 0 - 0 + 0.230 = 0.230 \\ Z_{12,23} &= Z_{12} - Z_{22} - Z_{13} + Z_{23} \\ &= 0 - 0.271 - 0 + 0.126 = -0.145 \end{aligned}$$

The elements of the remaining rows are obtained in a similar manner. The node-pair impedance matrix is

$$Z_{NP} = \begin{array}{c|cccc|cc} & 1\text{-}2 & 1\text{-}3 & 2\text{-}4 & 3\text{-}4 & 1\text{-}4 & 2\text{-}3 \\ \hline 1\text{-}2 & 0.271 & 0.126 & -0.041 & 0.104 & 0.230 & -0.145 \\ 1\text{-}3 & 0.126 & 0.344 & 0.063 & -0.155 & 0.189 & 0.218 \\ 2\text{-}4 & -0.041 & 0.063 & 0.173 & 0.069 & 0.132 & 0.104 \\ 3\text{-}4 & 0.104 & -0.155 & 0.069 & 0.328 & 0.173 & -0.259 \\ \hline 1\text{-}4 & 0.230 & 0.189 & 0.132 & 0.173 & 0.362 & -0.041 \\ 2\text{-}3 & -0.145 & 0.218 & 0.104 & -0.259 & -0.041 & 0.363 \end{array}$$

Then

$$Z_{p\sigma,\mu\nu} = \begin{array}{c|ccccc} & 1\text{-}2(1) & 1\text{-}2(2) & 1\text{-}3 & 2\text{-}4 & 3\text{-}4 \\ \hline 1\text{-}2(1) & 0.271 & 0.271 & 0.126 & -0.041 & 0.104 \\ 1\text{-}2(2) & 0.271 & 0.271 & 0.126 & -0.041 & 0.104 \\ 1\text{-}3 & 0.126 & 0.126 & 0.344 & 0.063 & -0.155 \\ 2\text{-}4 & -0.041 & -0.041 & 0.063 & 0.173 & 0.069 \\ 3\text{-}4 & 0.104 & 0.104 & -0.155 & 0.069 & 0.328 \end{array}$$

Second, form the submatrix Y_4 of the element-pair admittance matrix Y_{EP}. The elements of the submatrix are obtained from the equation

$$Y_{ij,pq} = y_{ij,pq} - \bar{y}_{ij,p\sigma}[Z_{p\sigma,\mu\nu}]\bar{y}_{\mu\nu,pq}$$

The element $Y_{12(2),12(2)}$ of Y_4 is

$$Y_{12(2),12(2)} = y_{12(2),12(2)} - \begin{bmatrix} y_{12(2),12(1)} & y_{12(2),12(2)} & y_{12(2),13} & y_{12(2),24} & y_{12(2),34} \end{bmatrix} [Z_{\rho\sigma,\mu\nu}] \begin{bmatrix} y_{12(1),12(2)} \\ y_{12(2),12(2)} \\ y_{13,12(2)} \\ y_{24,12(2)} \\ y_{34,12(2)} \end{bmatrix}$$

$$= 3.021 - \begin{bmatrix} -1.042 & 3.021 & 0.208 & & \end{bmatrix} \begin{bmatrix} 0.271 & 0.271 & 0.126 & -0.041 & 0.104 \\ 0.271 & 0.271 & 0.126 & -0.041 & 0.104 \\ 0.126 & 0.126 & 0.344 & 0.063 & -0.155 \\ -0.041 & -0.041 & 0.063 & 0.173 & 0.069 \\ 0.104 & 0.104 & -0.155 & 0.069 & 0.328 \end{bmatrix} \begin{bmatrix} -1.042 \\ 3.021 \\ 0.208 \\ \\ \end{bmatrix}$$

$$= 3.021 - \begin{bmatrix} 0.5625 & 0.5625 & 0.3209 & -0.0680 & 0.1736 \end{bmatrix} \begin{bmatrix} -1.042 \\ 3.021 \\ 0.208 \\ \\ \end{bmatrix}$$

$$= 3.021 - 1.180 = 1.841$$

The element $Y_{12(2),24}$ is

$$Y_{12(2),24} = y_{12(2),24} - \begin{bmatrix} y_{12(2),12(1)} & y_{12(2),12(2)} & y_{12(2),13} & y_{12(2),24} & y_{12(2),34} \end{bmatrix} [Z_{\rho\sigma,\mu\nu}] \begin{bmatrix} y_{12(1),24} \\ y_{12(2),24} \\ y_{13,24} \\ y_{24,24} \\ y_{34,24} \end{bmatrix}$$

$$= 0 - \begin{bmatrix} -1.042 & 3.021 & 0.208 & & \end{bmatrix} \begin{bmatrix} 0.271 & 0.271 & 0.126 & -0.041 & 0.104 \\ 0.271 & 0.271 & 0.126 & -0.041 & 0.104 \\ 0.126 & 0.126 & 0.344 & 0.063 & -0.155 \\ -0.041 & -0.041 & 0.063 & 0.173 & 0.069 \\ 0.104 & 0.104 & -0.155 & 0.069 & 0.328 \end{bmatrix} \begin{bmatrix} \\ \\ \\ 5.0 \\ \\ \end{bmatrix}$$

$$= 0 - (-0.340) = 0.340$$

The elements $Y_{24,12(2)}$ and $Y_{24,24}$ are calculated in a similar manner. The submatrix Y_4 is the loop admittance matrix

$$Y_{LOOP} = \begin{array}{c|c|c|} & D & E \\ \hline D & 1.841 & 0.340 \\ \hline E & 0.340 & 0.675 \\ \hline \end{array}$$

Problems

4.1 Using the data for the sample power system given in Prob. 3.2 and neglecting resistance, form the following positive sequence matrices:
   a. The bus impedance matrix using the algorithm
   b. The node-pair impedance matrix Z_{NP}
   c. The element-pair admittance matrix Y_{EP}
   d. The loop admittance matrix Y_{LOOP} from Y_{EP}

4.2 Repeat Prob. 4.1 using the zero sequence network data and neglecting resistance.

4.3 Modify the positive and zero sequence bus impedance matrices obtained in Probs. 4.1 and 4.2 to reflect the opening of the north circuit N between buses B and C.

4.4 Derive the equation

$$Z_{ij,pq} = z_{ij,pq} - \bar{z}_{ij,\rho\sigma}[Y_{\rho\sigma,\mu\nu}]\bar{z}_{\mu\nu,pq}$$

for obtaining the elements of the node-pair impedance matrix Z_{NP} using the element-pair admittance matrix Y_{EP}.

4.5 Form the node-pair impedance matrix Z_{NP} using the loop admittance matrix Y_{LOOP} obtained from Prob. 4.1, part *d*.

4.6 Show that the branch impedance matrix Z_{BR} can be derived from the node-pair impedance matrix Z_{NP}.

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chapter 5

Three-phase networks

5.1 Introduction

Power systems are operated usually with balanced three-phase generation and loads. A balanced network is obtained by the transposing of transmission lines. This makes possible the treatment of many three-phase power system problems on a single-phase basis. If there is unbalanced excitation on a balanced network the solution of network problems can be obtained by one of two methods. The first method analyzes the network in terms of actual phase quantities. The second method involves the transformation of unbalanced phase quantities into balanced sequence quantities. Two important types of sequence quantities are symmetrical components and Clarke's components. For symmetrical components, the balanced sequence impedances are uncoupled for both stationary and rotating elements. For Clarke's components the balanced sequence impedances are uncoupled only for stationary elements. Transformations for unbalanced networks, in general, do not yield uncoupled sequence impedances.

5.2 Three-phase network elements

A three-phase network component represented in impedance form is shown in Fig. 5.1. This component represented in admittance form is shown in Fig. 5.2. The variables and parameters are:

$v^a_{pq}, v^b_{pq}, v^c_{pq}$	are the voltages across the element p–q for phases a, b, and c, respectively
$e^a_{pq}, e^b_{pq}, e^c_{pq}$	are the source voltages in series with phases a, b, and c, respectively, of the element p–q

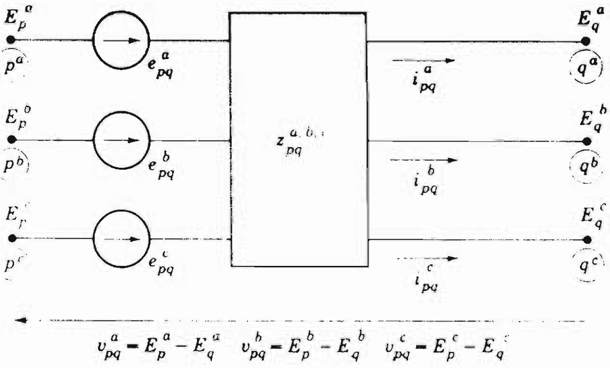

Fig. 5.1 ***Representation of three-phase network component in impedance form.***

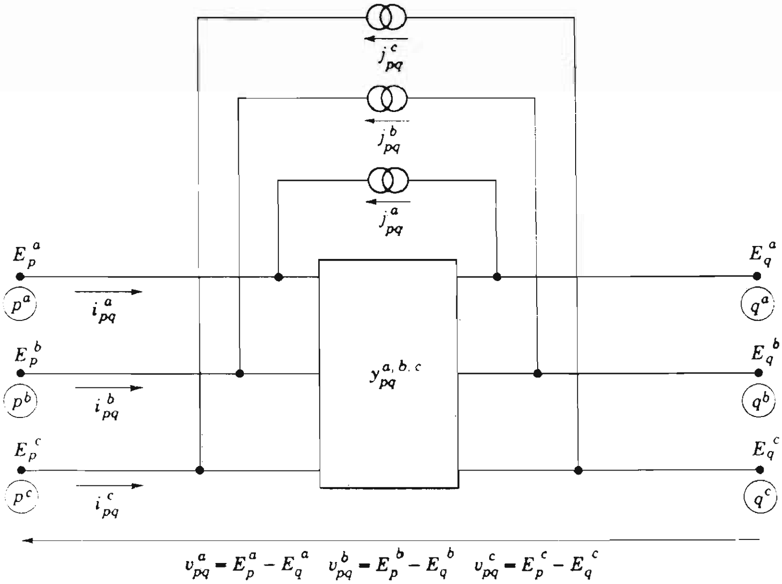

Fig. 5.2 ***Representation of three-phase network component in admittance form.***

$i_{pq}^{a}, i_{pq}^{b}, i_{pq}^{c}$ are the currents through the element p–q for phases a, b, and c, respectively

$j_{pq}^{a}, j_{pq}^{b}, j_{pq}^{c}$ are the source currents in parallel with phases a, b, and c, respectively, of the element p–q

$z_{pq}^{a,b,c}$ is the three-phase impedance matrix for the element p–q

$y_{pq}^{a,b,c}$ is the three-phase admittance matrix for the element p–q

The performance equation of a three-phase element in impedance form is

$$\begin{bmatrix} i_{pq}^{a} \\ i_{pq}^{b} \\ i_{pq}^{c} \end{bmatrix} + \begin{bmatrix} e_{pq}^{a} \\ e_{pq}^{b} \\ e_{pq}^{c} \end{bmatrix} = \begin{bmatrix} z_{pq}^{aa} & z_{pq}^{ab} & z_{pq}^{ac} \\ z_{pq}^{ba} & z_{pq}^{bb} & z_{pq}^{bc} \\ z_{pq}^{ca} & z_{pq}^{cb} & z_{pq}^{cc} \end{bmatrix} \begin{bmatrix} i_{pq}^{a} \\ i_{pq}^{b} \\ i_{pq}^{c} \end{bmatrix} \tag{5.2.1}$$

where z_{pq}^{aa} = self-impedance of phase a of the three-phase element connecting nodes p and q

z_{pq}^{ab} = mutual impedance between phases a and b

z_{pq}^{ac} = mutual impedance between phases a and c

and so forth.

Equation (5.2.1) can be written more concisely as

$$i_{pq}^{a,b,c} + e_{pq}^{a,b,c} = z_{pq}^{a,b,c} i_{pq}^{a,b,c} \tag{5.2.2}$$

The performance equation in admittance form is

$$\begin{bmatrix} i_{pq}^{a} \\ i_{pq}^{b} \\ i_{pq}^{c} \end{bmatrix} + \begin{bmatrix} j_{pq}^{a} \\ j_{pq}^{b} \\ j_{pq}^{c} \end{bmatrix} = \begin{bmatrix} y_{pq}^{aa} & y_{pq}^{ab} & y_{pq}^{ac} \\ y_{pq}^{ba} & y_{pq}^{bb} & y_{pq}^{bc} \\ y_{pq}^{ca} & y_{pq}^{cb} & y_{pq}^{cc} \end{bmatrix} \begin{bmatrix} v_{pq}^{a} \\ v_{pq}^{b} \\ v_{pq}^{c} \end{bmatrix}$$

which can be written

$$i_{pq}^{a,b,c} + j_{pq}^{a,b,c} = y_{pq}^{a,b,c} v_{pq}^{a,b,c}$$

where

$$y_{pq}^{a,b,c} = (z_{pq}^{a,b,c})^{-1}$$

The parallel three-phase source current in admittance form and the three-phase series source voltage in impedance form have the relationship, as is the case in single-phase representation,

$$j_{pq}^{a,b,c} = -y_{pq}^{a,b,c} e_{pq}^{a,b,c}$$

The impedance matrix $z_{pq}^{a,b,c}$ and the admittance matrix $y_{pq}^{a,b,c}$ of a stationary bilateral element are symmetric. If, in addition, the three-phase element is balanced, then the diagonal elements of $z_{pq}^{a,b,c}$, designated by z_{pq}^{s}, are equal and the off-diagonal elements, designated by z_{pq}^{m}, are equal, that is,

$$z_{pq}^{aa} = z_{pq}^{bb} = z_{pq}^{cc} = z_{pq}^{s}$$

and

$$z_{pq}^{ab} = z_{pq}^{ac} = z_{pq}^{ba} = z_{pq}^{bc} = z_{pq}^{ca} = z_{pq}^{cb} = z_{pq}^{m}$$

The corresponding relations are true in the admittance matrix $y_{pq}^{a,b,c}$.

The impedance and admittance matrices of balanced three-phase rotating elements are not symmetric. However, the mutual coupling from phase a to phase b, b to c, and c to a for the phase sequence a, b, c are identical, that is,

$$z_{pq}^{ab} = z_{pq}^{bc} = z_{pq}^{ca} = z_{pq}^{m1}$$

Similarly,

$$z_{pq}^{ac} = z_{pq}^{ba} = z_{pq}^{cb} = z_{pq}^{m2}$$

The performance equation of the three-phase primitive network in impedance form is

$$\bar{v}^{a,b,c} + \bar{e}^{a,b,c} = [z^{a,b,c}]\bar{\imath}^{a,b,c}$$

or in the admittance form is

$$\bar{\imath}^{a,b,c} + \bar{\jmath}^{a,b,c} = [y^{a,b,c}]\bar{v}^{a,b,c}$$

The vectors representing the variables are composed of 3 × 1 submatrices corresponding to the variables of a particular three-phase network element. The parameter matrices are composed of 3 × 3 submatrices. These submatrices correspond to the self and mutual three-phase impedance or admittance matrices of the network elements.

5.3 Three-phase balanced network elements

Balanced excitation

The excitation of any three-phase element is balanced when the source voltages or source currents of all phases are equal in magnitude and displaced from each other by 120°. For balanced excitation,

$$e^{a,b,c}_{pq} = \begin{array}{|c|}\hline e^a_{pq} \\ \hline e^b_{pq} \\ \hline e^c_{pq} \\ \hline \end{array} = \begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array} e^a_{pq} \quad \text{and} \quad j^{a,b,c}_{pq} = \begin{array}{|c|}\hline j^a_{pq} \\ \hline j^b_{pq} \\ \hline j^c_{pq} \\ \hline \end{array} = \begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array} j^a_{pq}$$

where

$$a = e^{j(2\pi/3)} = -\tfrac{1}{2} + j\tfrac{1}{2}\sqrt{3}$$

It follows that $a^3 = 1$, $a^2 + a + 1 = 0$, and $a^* = a^2$. The phase voltages and phase currents are balanced if the excitation of a balanced three-phase element is balanced. Then, the performance equation, in impedance form, for a stationary element is

$$\begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array} v^a_{pq} + \begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array} e^a_{pq} = \begin{array}{|c|c|c|}\hline z^s_{pq} & z^m_{pq} & z^m_{pq} \\ \hline z^m_{pq} & z^s_{pq} & z^m_{pq} \\ \hline z^m_{pq} & z^m_{pq} & z^s_{pq} \\ \hline \end{array} \begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array} i^a_{pq} \qquad (5.3.1)$$

and for a rotating element is

$$\begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array} v^a_{pq} + \begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array} e^a_{pq} = \begin{array}{|c|c|c|}\hline z^s_{pq} & z^{m1}_{pq} & z^{m2}_{pq} \\ \hline z^{m2}_{pq} & z^s_{pq} & z^{m1}_{pq} \\ \hline z^{m1}_{pq} & z^{m2}_{pq} & z^s_{pq} \\ \hline \end{array} \begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array} i^a_{pq} \qquad (5.3.2)$$

Both sides of equation (5.3.1) can be premultiplied by the conjugate transpose of

$$\begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array}$$

that is,

1	a	a^2

to obtain

$$3v^a_{pq} + 3e^a_{pq} = 3(z^s_{pq} - z^m_{pq})i^a_{pq} \tag{5.3.3}$$

Dividing by 3, equation (5.3.3) becomes

$$v^a_{pq} + e^a_{pq} = (z^s_{pq} - z^m_{pq})i^a_{pq}$$

where $(z^s_{pq} - z^m_{pq})$ is the positive sequence impedance, which is designated by $z^{(1)}_{pq}$. Thus, a balanced three-phase element with balanced excitation can be treated as a single-phase element in network problems. The power in the element is equal to three times the power per phase.

In a similar manner, equation (5.3.2) can be reduced to

$$v^a_{pq} + e^a_{pq} = (z^s_{pq} + a^2 z^{m1}_{pq} + a z^{m2}_{pq})i^a_{pq}$$

where $z^s_{pq} + a^2 z^{m1}_{pq} + a z^{m2}_{pq}$ is the positive sequence impedance.

The performance equation, in admittance form, for a stationary element is

$$i^a_{pq} + j^a_{pq} = (y^s_{pq} - y^m_{pq})v^a_{pq}$$

and for a rotating element is

$$i^a_{pq} + j^a_{pq} = (y^s_{pq} + a^2 y^{m1}_{pq} + a y^{m2}_{pq})v^a_{pq}$$

Unbalanced excitation

When the excitation is unbalanced, the performance equation of a three-phase element can be reduced to three independent equations by diagonalizing the impedance matrix $z^{a,b,c}_{pq}$. Using a complex transformation matrix T then the phase variables are expressed in terms of a new set of variables as follows:

$$\begin{aligned} v^{a,b,c}_{pq} &= T v^{i,j,k}_{pq} \\ e^{a,b,c}_{pq} &= T e^{i,j,k}_{pq} \\ i^{a,b,c}_{pq} &= T i^{i,j,k}_{pq} \end{aligned} \tag{5.3.4}$$

The complex power in the element is

$$S_{pq} = P_{pq} + jQ_{pq} = \{(i^{a,b,c}_{pq})^*\}^t e^{a,b,c}_{pq}$$

Substituting from equations (5.3.4),

$$S_{pq} = \{(i^{i,j,k}_{pq})^*\}^t (T^*)^t T e^{i,j,k}_{pq} \tag{5.3.5}$$

The complex power in terms of the i, j, k sequence variables is

$$S'_{pq} = \{(i^{i,j,k}_{pq})^*\}^t e^{i,j,k}_{pq} \tag{5.3.6}$$

If the complex powers S_{pq} and S'_{pq} are equal, that is, the selected transformation T is power-invariant, then from equations (5.3.5) and (5.3.6),

$$(T^*)^t T = U = T(T^*)^t$$

Thus T is a unitary matrix.

Substituting from equations (5.3.4) the performance equation (5.2.2) becomes

$$T(v^{i,j,k}_{pq} + e^{i,j,k}_{pq}) = z^{a,b,c}_{pq} T i^{i,j,k}_{pq} \tag{5.3.7}$$

Both sides of equation (5.3.7) can be premultiplied by $(T^*)^t$ to obtain

$$v^{i,j,k}_{pq} + e^{i,j,k}_{pq} = (T^*)^t z^{a,b,c}_{pq} T i^{i,j,k}_{pq}$$

It follows that

$$z^{i,j,k}_{pq} = (T^*)^t z^{a,b,c}_{pq} T \tag{5.3.8}$$

A similar transformation can be obtained for the performance equation in its admittance form.

5.4 Transformation matrices

Symmetrical components

Two particular transformations for three-phase balanced elements are of interest. One of these transforms the three-phase quantities into zero, positive, and negative sequence quantities, known as symmetrical components. The matrix for this transformation is

$$T_s = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

which is a unitary matrix, that is, $(T_s^*)^t T_s = U$; and furthermore, because T_s is symmetric, $T_s^* = T_s^{-1}$. Using this transformation the impedance matrix for a stationary element $z_{pq}^{i,j,k}$ from equation (5.3.8) becomes

$$z_{pq}^{0,1,2} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} z_{pq}^{s} & z_{pq}^{m} & z_{pq}^{m} \\ z_{pq}^{m} & z_{pq}^{s} & z_{pq}^{m} \\ z_{pq}^{m} & z_{pq}^{m} & z_{pq}^{s} \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

That is,

$$z_{pq}^{0,1,2} = \begin{bmatrix} z_{pq}^{s} + 2z_{pq}^{m} & & \\ & z_{pq}^{s} - z_{pq}^{m} & \\ & & z_{pq}^{s} - z_{pq}^{m} \end{bmatrix}$$

where the zero sequence impedance is

$$z_{pq}^{(0)} = z_{pq}^{s} + 2z_{pq}^{m}$$

the positive sequence impedance is

$$z_{pq}^{(1)} = z_{pq}^{s} - z_{pq}^{m}$$

the negative sequence impedance is

$$z_{pq}^{(2)} = z_{pq}^{s} - z_{pq}^{m}$$

and $z_{pq}^{0,1,2}$ refers to the transformed impedance matrix, which is diagonal for a balanced three-phase element.

The transformation matrix T_s also diagonalizes the impedance matrix for a rotating element, even though $z_{pq}^{a,b,c}$ is not symmetric. This diagonalized matrix is

$$z_{pq}^{0,1,2} = \begin{bmatrix} z_{pq}^{s} + z_{pq}^{m1} + z_{pq}^{m2} & & \\ & z_{pq}^{s} + a^2 z_{pq}^{m1} + a z_{pq}^{m2} & \\ & & z_{pq}^{s} + a z_{pq}^{m1} + a^2 z_{pq}^{m2} \end{bmatrix}$$

where
$$\begin{aligned} z_{pq}^{(0)} &= z_{pq}^{s} + z_{pq}^{m1} + z_{pq}^{m2} \\ z_{pq}^{(1)} &= z_{pq}^{s} + a^2 z_{pq}^{m1} + a z_{pq}^{m2} \\ z_{pq}^{(2)} &= z_{pq}^{s} + a z_{pq}^{m1} + a^2 z_{pq}^{m2} \end{aligned}$$

Clarke's components

Another transformation matrix transforms the three-phase quantities into zero, alpha, and beta sequence quantities, known as Clarke's components. The matrix for this transformation is

$$T_c = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 1 & -\sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} \\ 1 & -\sqrt{\frac{1}{2}} & -\sqrt{\frac{3}{2}} \end{bmatrix}$$

which is an orthogonal matrix, that is, $T_c^t T_c = U$. Therefore $T_c^t = T_c^{-1}$. Using this transformation the impedance matrix for a stationary element $z_{pq}^{i,j,k}$ from equation (5.3.8) becomes

$$z_{pq}^{0,\alpha,\beta} = \begin{bmatrix} z_{pq}^s + 2z_{pq}^m & & \\ & z_{pq}^s - z_{pq}^m & \\ & & z_{pq}^s - z_{pq}^m \end{bmatrix}$$

where the diagonal elements are the zero, alpha, and beta impedance components, respectively, and $z_{pq}^{0,\alpha,\beta}$ refers to the transformed impedance matrix which is diagonal for a balanced three-phase element.

The transformation matrix T_c does not diagonalize the nonsymmetric impedance matrix $z_{pq}^{a,b,c}$ for a rotating element. The following is obtained by this transformation.

$$z_{pq}^{0,\alpha,\beta} = \begin{bmatrix} z_{pq}^s + z_{pq}^{m1} + z_{pq}^{m2} & & \\ & z_{pq}^s - \frac{1}{2}(z_{pq}^{m1} + z_{pq}^{m2}) & \frac{\sqrt{3}}{2}(z_{pq}^{m1} - z_{pq}^{m2}) \\ & \frac{\sqrt{3}}{2}(z_{pq}^{m2} - z_{pq}^{m1}) & z^s - \frac{1}{2}(z_{pq}^{m1} + z_{pq}^{m2}) \end{bmatrix}$$

5.5 Three-phase unbalanced network elements

When a three-phase element is unbalanced, the transformation T_s, or T_c, on $z_{pq}^{a,b,c}$ does not yield uncoupled sequence impedances. Even though it is possible to diagonalize $z_{pq}^{a,b,c}$, no single transformation exists for diagonalizing the impedance matrices for all elements of a network because the unbalance of the different elements, in general, is not related. Therefore, it may be desirable to maintain the original three-phase quantities for the solution of network problems. When the transformation T_s is used the sequence networks cannot be treated independently.

5.6 Incidence and network matrices for three-phase networks

Incidence and network matrices for a three-phase balanced or unbalanced network can be formed by the same procedures as those described in Chap. 3 for single-phase networks. The entries 1, -1, and 0 in the incidence matrices for a single-phase network, however, will be replaced by the 3×3 matrices, U, $-U$, and null, respectively. Also, the impedance or admittance of a network element will be a 3×3 matrix. The rows and columns of this matrix refer to the phases a, b, and c or to the appropriate sequence components. The network matrices will be composed of 3×3 submatrices whose elements also refer to the phase or sequence components.

5.7 Algorithm for formation of three-phase bus impedance matrix

Performance equation of a partial three-phase network

The performance equation for a three-phase network representation in the bus frame of reference and impedance form is

$$\bar{E}_{BUS}^{a,b,c} = Z_{BUS}^{a,b,c} \bar{I}_{BUS}^{a,b,c}$$

where $\bar{E}_{BUS}^{a,b,c}$ = vector of the three-phase bus voltages measured with respect to the reference bus

$\bar{I}_{BUS}^{a,b,c}$ = vector of impressed three-phase bus currents

$Z_{BUS}^{a,b,c}$ = three-phase bus impedance matrix

When the three-phase elements of the network are balanced, their impedance or admittance matrices can be diagonalized by the transformation matrix T_s or T_c. In this case, the three sequence networks can be treated independently. The procedures based on the algorithm described in Chap. 4 can be applied to form the independent sequence network matrices.

When the three-phase elements of the network are unbalanced, the 3×3 submatrices $Z_{ij}^{a,b,c}$ and $Z_{ji}^{a,b,c}$ of the bus impedance matrix are not equal. The equations for the formation of the three-phase bus impedance matrix by the algorithm can be derived in a manner similar to that for single-phase networks.

Addition of a branch

The performance equation of the partial network with an added branch p–q, in terms of three-phase quantities, is

$$\begin{array}{c} \\ E_1^{a,b,c}\\ E_2^{a,b,c}\\ \cdots\\ E_p^{a,b,c}\\ \cdots\\ E_m^{a,b,c}\\ \hline E_q^{a,b,c}\end{array}
\begin{array}{c} \\ \\ \\ \\ = \\ \\ \\ \\ \end{array}
\begin{array}{c|ccccccc|c}
 & 1 & 2 & & p & & m & & q\\
\hline
1 & Z_{11}^{a,b,c} & Z_{12}^{a,b,c} & \cdots & Z_{1p}^{a,b,c} & \cdots & Z_{1m}^{a,b,c} & & Z_{1q}^{a,b,c}\\
2 & Z_{21}^{a,b,c} & Z_{22}^{a,b,c} & \cdots & Z_{2p}^{a,b,c} & \cdots & Z_{2m}^{a,b,c} & & Z_{2q}^{a,b,c}\\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots\\
p & Z_{p1}^{a,b,c} & Z_{p2}^{a,b,c} & \cdots & Z_{pp}^{a,b,c} & \cdots & Z_{pm}^{a,b,c} & & Z_{pq}^{a,b,c}\\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots\\
m & Z_{m1}^{a,b,c} & Z_{m2}^{a,b,c} & \cdots & Z_{mp}^{a,b,c} & \cdots & Z_{mm}^{a,b,c} & & Z_{mq}^{a,b,c}\\
\hline
q & Z_{q1}^{a,b,c} & Z_{q2}^{a,b,c} & \cdots & Z_{qp}^{a,b,c} & \cdots & Z_{qm}^{a,b,c} & & Z_{qq}^{a,b,c}
\end{array}
\begin{array}{c} \\ I_1^{a,b,c}\\ I_2^{a,b,c}\\ \cdots\\ I_p^{a,b,c}\\ \cdots\\ I_m^{a,b,c}\\ \hline I_q^{a,b,c}\end{array}
\qquad (5.7.1)$$

The elements $Z_{qi}^{a,b,c}$ can be determined by injecting a three-phase current at the ith bus, as shown in Fig. 5.3, and measuring the voltage at the qth bus with respect to the reference node. Similarly, the elements $Z_{iq}^{a,b,c}$ can be determined by injecting a three-phase current at the qth bus, as

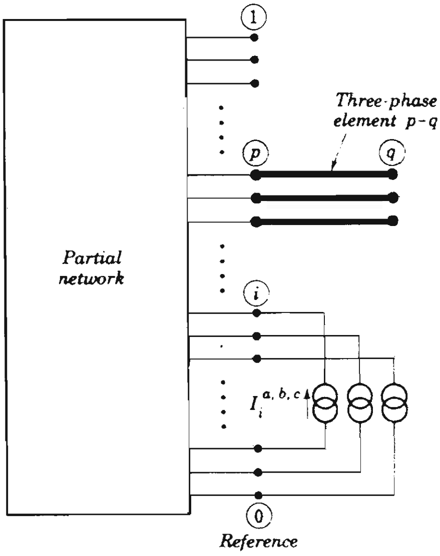

Fig. 5.3 Injected three-phase current for calculation of $Z_{qi}^{a,b,c}$.

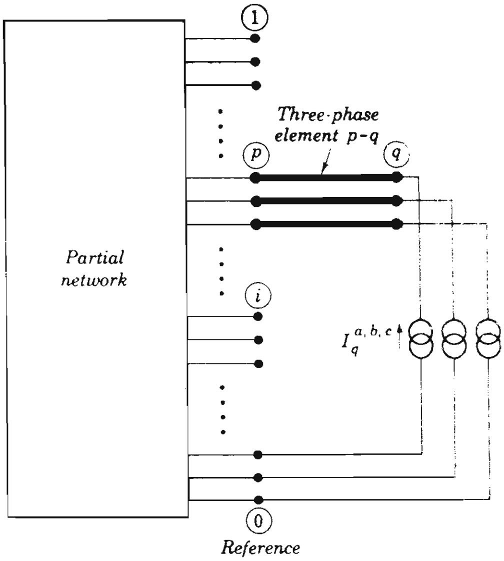

Fig. 5.4 *Injected three-phase current for calculation of $Z_{iq}^{a,b,c}$ and $Z_{qq}^{a,b,c}$.*

shown in Fig. 5.4, and measuring the voltage at the ith bus with respect to the reference node.

To calculate $Z_{qi}^{a,b,c}$ let the current at the ith bus be $I_i^{a,b,c}$ and all other bus currents equal zero. The voltage across the added element p–q is

$$v_{pq}^{a,b,c} = E_p^{a,b,c} - E_q^{a,b,c} \tag{5.7.2}$$

The vector of voltages across the elements ρ–σ of the partial network is

$$\bar{v}_{\rho\sigma}^{a,b,c} = \bar{E}_\rho^{a,b,c} - \bar{E}_\sigma^{a,b,c} \tag{5.7.3}$$

The current in the element p–q, in terms of the primitive admittances and the voltages across the elements, is

$$i_{pq}^{a,b,c} = y_{pq,pq}^{a,b,c} v_{pq}^{a,b,c} + \bar{y}_{pq,\rho\sigma}^{a,b,c} \bar{v}_{\rho\sigma}^{a,b,c} \tag{5.7.4}$$

Since $i_{pq}^{a,b,c} = 0$, from equation (5.7.4),

$$v_{pq}^{a,b,c} = -(y_{pq,pq}^{a,b,c})^{-1} \bar{y}_{pq,\rho\sigma}^{a,b,c} \bar{v}_{\rho\sigma}^{a,b,c} \tag{5.7.5}$$

Substituting from equations (5.7.2) and (5.7.3) into (5.7.5),

$$E_p^{a,b,c} - E_q^{a,b,c} = -(y_{pq,pq}^{a,b,c})^{-1} \bar{y}_{pq,\rho\sigma}^{a,b,c} (\bar{E}_\rho^{a,b,c} - \bar{E}_\sigma^{a,b,c}) \tag{5.7.6}$$

From equation (5.7.1),

$$\begin{aligned} E_p^{a,b,c} &= Z_{pi}^{a,b,c} I_i^{a,b,c} \\ E_q^{a,b,c} &= Z_{qi}^{a,b,c} I_i^{a,b,c} \end{aligned} \tag{5.7.7}$$

and at any bus k,

$$E_k^{a,b,c} = Z_{ki}^{a,b,c} I_i^{a,b,c}$$

Using the relationships from equation (5.7.7) in (5.7.6) and solving for $Z_{qi}^{a,b,c} I_i^{a,b,c}$,

$$Z_{qi}^{a,b,c} I_i^{a,b,c} = Z_{pi}^{a,b,c} I_i^{a,b,c} + \{(y_{pq,pq}^{a,b,c})^{-1} \bar{y}_{pq,\rho\sigma}^{a,b,c} (\bar{Z}_{\rho i}^{a,b,c} - \bar{Z}_{\sigma i}^{a,b,c})\} I_i^{a,b,c} \tag{5.7.8}$$

Since equation (5.7.8) is valid for all values of $I_i^{a,b,c}$, it follows that

$$Z_{qi}^{a,b,c} = Z_{pi}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1} \bar{y}_{pq,\rho\sigma}^{a,b,c} (\bar{Z}_{\rho i}^{a,b,c} - \bar{Z}_{\sigma i}^{a,b,c}) \tag{5.7.9}$$

To calculate $Z_{iq}^{a,b,c}$, let the current at the qth bus be $I_q^{a,b,c}$ and all other bus currents equal zero. If the added element p–q were not mutually coupled to the elements of the partial network, the voltage at the ith bus would be the same whether the current $I_q^{a,b,c}$ is injected at bus p or q, that is, $I_p^{a,b,c} = I_q^{a,b,c}$, and therefore

$$E_i^{a,b,c} = Z_{iq}^{a,b,c} I_q^{a,b,c} = Z_{ip}^{a,b,c} I_q^{a,b,c}$$

However, the element p–q is assumed to be mutually coupled to one or more elements of the partial network; therefore,

$$E_i^{a,b,c} = Z_{iq}^{a,b,c} I_q^{a,b,c} = Z_{ip}^{a,b,c} I_q^{a,b,c} + \Delta E_i^{a,b,c} \tag{5.7.10}$$

where $\Delta E_i^{a,b,c}$ is the change in voltage at bus i due to the effect of mutual coupling. The vector of voltages induced in the elements ρ–σ is

$$\bar{e}_{\rho\sigma}^{a,b,c} = \bar{z}_{\rho\sigma,pq}^{a,b,c} I_q^{a,b,c} \tag{5.7.11}$$

The series source voltages $\bar{e}_{\rho\sigma}^{a,b,c}$ and parallel source currents $\bar{j}_{\rho\sigma}^{a,b,c}$ are related by

$$\bar{j}_{\rho\sigma}^{a,b,c} = -[z_{\rho\sigma,\rho\sigma}^{a,b,c}]^{-1} \bar{e}_{\rho\sigma}^{a,b,c} \tag{5.7.12}$$

Substituting from equation (5.7.11) into (5.7.12),

$$\bar{j}_{\rho\sigma}^{a,b,c} = -[z_{\rho\sigma,\rho\sigma}^{a,b,c}]^{-1} \bar{z}_{\rho\sigma,pq}^{a,b,c} I_q^{a,b,c} \tag{5.7.13}$$

However, in terms of bus currents,

$$\bar{j}_{\rho\sigma}^{a,b,c} = \bar{I}_\rho^{a,b,c} = -\bar{I}_\sigma^{a,b,c} \tag{5.7.14}$$

and the change in voltage at the ith bus due to mutual coupling is

$$\Delta E_i^{a,b,c} = \bar{Z}_{i\rho}^{a,b,c} \bar{I}_\rho^{a,b,c} + \bar{Z}_{i\sigma}^{a,b,c} \bar{I}_\sigma^{a,b,c} \tag{5.7.15}$$

Substituting for $\bar{I}_\rho^{a,b,c}$ and $\bar{I}_\sigma^{a,b,c}$, from equations (5.7.13) and (5.7.14), equation (5.7.15) becomes

$$\Delta E_i^{a,b,c} = -(\bar{Z}_{i\rho}^{a,b,c} - \bar{Z}_{i\sigma}^{a,b,c}) [z_{\rho\sigma,\rho\sigma}^{a,b,c}]^{-1} \bar{z}_{\rho\sigma,pq}^{a,b,c} I_q^{a,b,c} \tag{5.7.16}$$

Substituting from equation (5.7.16) into (5.7.10), it follows that

$$Z_{iq}^{a,b,c} = Z_{ip}^{a,b,c} - (\bar{Z}_{ip}^{a,b,c} - \bar{Z}_{i\sigma}^{a,b,c})[z_{\rho\sigma,\rho\sigma}^{a,b,c}]^{-1}\bar{z}_{\rho\sigma,pq}^{a,b,c} \tag{5.7.17}$$

From the matrix equation,

$$\left[\begin{array}{c|c} z_{pq,pq}^{a,b,c} & \bar{z}_{pq,\rho\sigma}^{a,b,c} \\ \hline \bar{z}_{\rho\sigma,pq}^{a,b,c} & z_{\rho\sigma,\rho\sigma}^{a,b,c} \end{array}\right] \left[\begin{array}{c|c} y_{pq,pq}^{a,b,c} & \bar{y}_{pq,\rho\sigma}^{a,b,c} \\ \hline \bar{y}_{\rho\sigma,pq}^{a,b,c} & y_{\rho\sigma,\rho\sigma}^{a,b,c} \end{array}\right] = \left[\begin{array}{c|c} U & \\ \hline & U \end{array}\right]$$

then

$$\bar{z}_{\rho\sigma,pq}^{a,b,c} y_{pq,pq}^{a,b,c} = -[z_{\rho\sigma,\rho\sigma}^{a,b,c}]\bar{y}_{\rho\sigma,pq}^{a,b,c} \tag{5.7.18}$$

Premultiplying by $[z_{\rho\sigma,\rho\sigma}^{a,b,c}]^{-1}$ and postmultiplying by $(y_{pq,pq}^{a,b,c})^{-1}$, equation (5.7.18) becomes

$$[z_{\rho\sigma,\rho\sigma}^{a,b,c}]^{-1}\bar{z}_{\rho\sigma,pq}^{a,b,c} = -\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1} \tag{5.7.19}$$

Substituting from equation (5.7.19) into (5.7.17),

$$Z_{iq}^{a,b,c} = Z_{ip}^{a,b,c} + (\bar{Z}_{ip}^{a,b,c} - \bar{Z}_{i\sigma}^{a,b,c})\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1} \tag{5.7.20}$$

The element $Z_{qq}^{a,b,c}$ can be determined by injecting a three-phase current at the qth bus and measuring the voltage at that bus with respect to the reference node. Let the current at the qth bus be $I_q^{a,b,c}$ and all other bus currents equal zero. Since $i_{pq}^{a,b,c} = -I_q^{a,b,c}$, substituting in equation (5.7.4) for $i_{pq}^{a,b,c}$ and solving for $v_{pq}^{a,b,c}$,

$$v_{pq}^{a,b,c} = -(y_{pq,pq}^{a,b,c})^{-1}(I_q^{a,b,c} + \bar{y}_{pq,\rho\sigma}^{a,b,c}\bar{v}_{\rho\sigma}^{a,b,c}) \tag{5.7.21}$$

Substituting from equations (5.7.2) and (5.7.3) into (5.7.21),

$$E_p^{a,b,c} - E_q^{a,b,c} = -(y_{pq,pq}^{a,b,c})^{-1}\{I_q^{a,b,c} + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\bar{E}_\rho^{a,b,c} - \bar{E}_\sigma^{a,b,c})\} \tag{5.7.22}$$

From equation (5.7.1),

$$\begin{aligned} E_p^{a,b,c} &= Z_{pq}^{a,b,c} I_q^{a,b,c} \\ E_q^{a,b,c} &= Z_{qq}^{a,b,c} I_q^{a,b,c} \end{aligned} \tag{5.7.23}$$

and at any bus k,

$$E_k^{a,b,c} = Z_{kq}^{a,b,c} I_q^{a,b,c}$$

Substituting from equation (5.7.23) into (5.7.22) and solving for $Z_{qq}^{a,b,c} I_q^{a,b,c}$, it follows, since the resulting equation is valid for all values of $I_q^{a,b,c}$, that

$$Z_{qq}^{a,b,c} = Z_{pq}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\{U + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\bar{Z}_{\rho q}^{a,b,c} - \bar{Z}_{\sigma q}^{a,b,c})\} \tag{5.7.24}$$

If there is no mutual coupling between the added branch and the elements of the partial network, the elements of $\bar{y}^{a,b,c}_{pq,\rho\sigma}$ are zero and $(y^{a,b,c}_{pq,pq})^{-1} = z^{a,b,c}_{pq,pq}$ Then, equations (5.7.9), (5.7.20), and (5.7.24) reduce to

$$
\begin{aligned}
Z^{a,b,c}_{qi} &= Z^{a,b,c}_{pi}\\
Z^{a,b,c}_{iq} &= Z^{a,b,c}_{ip}\\
Z^{a,b,c}_{qq} &= Z^{a,b,c}_{pq} + z^{a,b,c}_{pq,pq}
\end{aligned}
$$

If, in addition, p is the reference node, the elements of $Z^{a,b,c}_{qi}$ and $Z^{a,b,c}_{iq}$ are zero. Also

$$Z^{a,b,c}_{qq} = z^{a,b,c}_{pq,pq}$$

If the network elements are balanced, then $Z^{a,b,c}_{qi} = Z^{a,b,c}_{iq}$ and either equation (5.7.9) or (5.7.20) can be used.

Addition of a link

As in the case of single-phase networks, when the new element is a link it is connected in series with a voltage source as shown in Fig. 5.5. The three-phase voltage source $e^{a,b,c}_l$ is selected such that the current through the added link is zero. Then the element p–l, where l is a fictitious node,

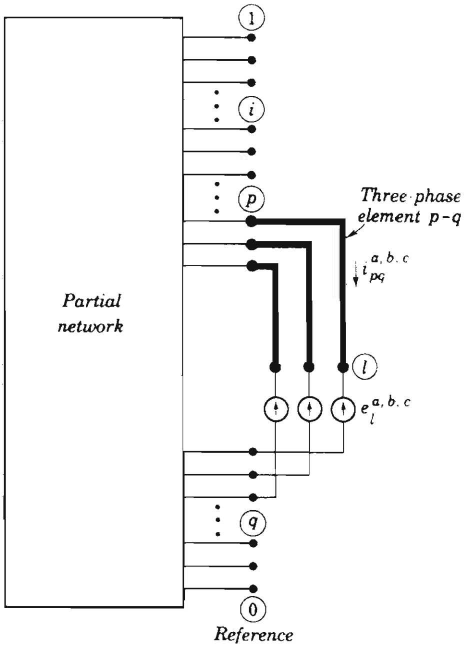

Fig. 5.5 *Three-phase voltage source in series with added link for calculation of $Z^{a,b,c}_{li}$, $Z^{a,b,c}_{il}$, and $Z^{a,b,c}_{ll}$.*

can be treated as a branch. The performance equation of the partial network with the added branch p-l, in terms of three-phase quantities, is

$$\begin{array}{c} \\ E_1^{a,b,c} \\ E_2^{a,b,c} \\ \cdots \\ E_p^{a,b,c} \\ \cdots \\ E_m^{a,b,c} \\ \hline e_l^{a,b,c} \end{array} \begin{array}{c} \\ \\ \\ \\ = \\ \\ \\ \\ \end{array} \begin{array}{c|ccccccc|c} & 1 & 2 & & p & & m & l \\ \hline 1 & Z_{11}^{a,b,c} & Z_{12}^{a,b,c} & \cdots & Z_{1p}^{a,b,c} & \cdots & Z_{1m}^{a,b,c} & Z_{1l}^{a,b,c} \\ 2 & Z_{21}^{a,b,c} & Z_{22}^{a,b,c} & \cdots & Z_{2p}^{a,b,c} & \cdots & Z_{2m}^{a,b,c} & Z_{2l}^{a,b,c} \\ & \cdots & & & & & \cdots & \\ p & Z_{p1}^{a,b,c} & Z_{p2}^{a,b,c} & \cdots & Z_{pp}^{a,b,c} & \cdots & Z_{pm}^{a,b,c} & Z_{pl}^{a,b,c} \\ & \cdots & \cdots & & \cdots & & \cdots & \cdots \\ m & Z_{m1}^{a,b,c} & Z_{m2}^{a,b,c} & \cdots & Z_{mp}^{a,b,c} & \cdots & Z_{mm}^{a,b,c} & Z_{ml}^{a,b,c} \\ \hline l & Z_{l1}^{a,b,c} & Z_{l2}^{a,b,c} & \cdots & Z_{lp}^{a,b,c} & \cdots & Z_{lm}^{a,b,c} & Z_{ll}^{a,b,c} \end{array} \begin{array}{c} \\ I_1^{a,b,c} \\ I_2^{a,b,c} \\ \cdots \\ I_p^{a,b,c} \\ \cdots \\ I_m^{a,b,c} \\ \hline I_l^{a,b,c} \end{array} \tag{5.7.25}$$

The elements $Z_{li}^{a,b,c}$ can be determined by injecting a three-phase current at the ith bus and measuring the voltage at the fictitious node l with respect to bus q. Let the current at the ith bus be $I_i^{a,b,c}$ and all other bus currents equal zero. Then, from equation (5.7.25),

$$e_l^{a,b,c} = Z_{li}^{a,b,c} I_i^{a,b,c} \tag{5.7.26}$$

Also, as shown in Fig. 5.5,

$$e_l^{a,b,c} = v_{pq}^{a,b,c} - v_{pl}^{a,b,c} \tag{5.7.27}$$

The current $i_{pq}^{a,b,c}$ in terms of the primitive admittances and the voltages across the elements is

$$i_{pq}^{a,b,c} = y_{pl,pl}^{a,b,c} v_{pl}^{a,b,c} + \bar{y}_{pl,\rho\sigma}^{a,b,c} \bar{v}_{\rho\sigma}^{a,b,c} \tag{5.7.28}$$

Since

$$\bar{y}_{pl,\rho\sigma}^{a,b,c} = \bar{y}_{pq,\rho\sigma}^{a,b,c}$$
$$y_{pl,pl}^{a,b,c} = y_{pq,pq}^{a,b,c}$$

and the elements of $i_{pq}^{a,b,c}$ are zero, then the voltage $v_{pl}^{a,b,c}$, from equation (5.7.28), is

$$v_{pl}^{a,b,c} = -(y_{pq,pq}^{a,b,c})^{-1} \bar{y}_{pq,\rho\sigma}^{a,b,c} \bar{v}_{\rho\sigma}^{a,b,c} \tag{5.7.29}$$

Substituting from equation (5.7.29) into (5.7.27),

$$e_l^{a,b,c} = v_{pq}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1} \bar{y}_{pq,\rho\sigma}^{a,b,c} \bar{v}_{\rho\sigma}^{a,b,c} \tag{5.7.30}$$

Substituting for $v_{pq}^{a,b,c}$ and $\bar{v}_{\rho\sigma}^{a,b,c}$ from equations (5.7.2), (5.7.3), and (5.7.7) into (5.7.30),

$$e_l^{a,b,c} = (Z_{pi}^{a,b,c} - Z_{qi}^{a,b,c})I_i^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\bar{y}_{pq,\rho\sigma}^{a,b,c}(\bar{Z}_{\rho i}^{a,b,c} - \bar{Z}_{\sigma i}^{a,b,c})I_i^{a,b,c} \tag{5.7.31}$$

Substituting for $e_l^{a,b,c}$ from equation (5.7.26) into (5.7.31), it follows, since the resulting equation is valid for all values of $I_i^{a,b,c}$, that

$$Z_{li}^{a,b,c} = Z_{pi}^{a,b,c} - Z_{qi}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\bar{y}_{pq,\rho\sigma}^{a,b,c}(\bar{Z}_{\rho i}^{a,b,c} - \bar{Z}_{\sigma i}^{a,b,c}) \tag{5.7.32}$$

The elements $Z_{il}^{a,b,c}$ can be determined by injecting a three-phase current between q and l and measuring the voltage at bus i. Let the current between q and l be $I_l^{a,b,c}$ and all other bus currents equal zero. If the added element were not mutually coupled to the elements of the partial network, then

$$E_i^{a,b,c} = (Z_{ip}^{a,b,c} - Z_{iq}^{a,b,c})I_l^{a,b,c}$$

However, because of the effect of mutual coupling,

$$E_i^{a,b,c} = Z_{il}^{a,b,c}I_l^{a,b,c} = (Z_{ip}^{a,b,c} - Z_{iq}^{a,b,c})I_i^{a,b,c} + \Delta E_i^{a,b,c} \tag{5.7.33}$$

Following the same procedure as in the case where the added element is a branch, then

$$\Delta E_i^{a,b,c} = -(\bar{Z}_{i\rho}^{a,b,c} - \bar{Z}_{i\sigma}^{a,b,c})[z_{\rho\sigma,\rho\sigma}^{a,b,c}]^{-1}\bar{z}_{\rho\sigma,pq}^{a,b,c}I_l^{a,b,c} \tag{5.7.34}$$

and from equation (5.7.19),

$$[z_{\rho\sigma,\rho\sigma}^{a,b,c}]^{-1}\bar{z}_{\rho\sigma,pq}^{a,b,c} = -\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1}$$

Substituting from equation (5.7.34) into equation (5.7.33) it follows, since the resultant equation is valid for all values of $I_l^{a,b,c}$, that

$$Z_{il}^{a,b,c} = Z_{ip}^{a,b,c} - Z_{iq}^{a,b,c} + (\bar{Z}_{i\rho}^{a,b,c} - \bar{Z}_{i\sigma}^{a,b,c})\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1} \tag{5.7.35}$$

The element $Z_{ll}^{a,b,c}$ can be determined by injecting a three-phase current between q and l and measuring the voltage at node l with respect to bus q. Let the current between q and l be $I_l^{a,b,c}$ and all other bus currents equal zero. Since $i_{pq}^{a,b,c} = -I_l^{a,b,c}$, substituting in equation (5.7.28) for $i_{pq}^{a,b,c}$ and solving for $v_{pl}^{a,b,c}$, then

$$v_{pl}^{a,b,c} = -(y_{pq,pq}^{a,b,c})^{-1}(I_l^{a,b,c} + \bar{y}_{pq,\rho\sigma}^{a,b,c}\bar{v}_{\rho\sigma}^{a,b,c}) \tag{5.7.36}$$

Substituting from equation (5.7.36) into (5.7.27),

$$e_l^{a,b,c} = v_{pq}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}(I_l^{a,b,c} + \bar{y}_{pq,\rho\sigma}^{a,b,c}\bar{v}_{\rho\sigma}^{a,b,c}) \tag{5.7.37}$$

Substituting for $v_{pq}^{a,b,c}$ and $\bar{v}_{\rho\sigma}^{a,b,c}$ from equations (5.7.2), (5.7.3), and (5.7.7) into (5.7.37),

$$e_l^{a,b,c} = (Z_{pl}^{a,b,c} - Z_{ql}^{a,b,c})I_l^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\{I_l^{a,b,c} + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\bar{Z}_{\rho l}^{a,b,c} - \bar{Z}_{\sigma l}^{a,b,c})I_l^{a,b,c}\} \tag{5.7.38}$$

Table 5.1 *Summary of equations for formation of three-phase bus impedance matrix*

Add	*Mutual coupling*	
p–q	*p is not the reference bus*	*p is the reference bus*
Branch	$Z_{qi}^{a,b,c} = Z_{pi}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\bar{y}_{pq,\rho\sigma}^{a,b,c}(\mathbf{Z}_{\rho i}^{a,b,c} - \mathbf{Z}_{\sigma i}^{a,b,c})$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{iq}^{a,b,c} = Z_{ip}^{a,b,c} + (\mathbf{Z}_{i\rho}^{a,b,c} - \mathbf{Z}_{i\sigma}^{a,b,c})\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1}$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{qq}^{a,b,c} = Z_{pq}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\{U + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\mathbf{Z}_{\rho q}^{a,b,c} - \mathbf{Z}_{\sigma q}^{a,b,c})\}$	$Z_{qi}^{a,b,c} = (y_{pq,pq}^{a,b,c})^{-1}\bar{y}_{pq,\rho\sigma}^{a,b,c}(\mathbf{Z}_{\rho i}^{a,b,c} - \mathbf{Z}_{\sigma i}^{a,b,c})$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{iq}^{a,b,c} = (\mathbf{Z}_{i\rho}^{a,b,c} - \mathbf{Z}_{i\sigma}^{a,b,c})\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1}$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{qq}^{a,b,c} = (y_{pq,pq}^{a,b,c})^{-1}\{U + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\mathbf{Z}_{\rho q}^{a,b,c} - \mathbf{Z}_{\sigma q}^{a,b,c})\}$
Link	$Z_{li}^{a,b,c} = Z_{pi}^{a,b,c} - Z_{qi}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\bar{y}_{pq,\rho\sigma}^{a,b,c}(\mathbf{Z}_{\rho i}^{a,b,c} - \mathbf{Z}_{\sigma i}^{a,b,c})$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{il}^{a,b,c} = Z_{ip}^{a,b,c} - Z_{iq}^{a,b,c} + (\mathbf{Z}_{i\rho}^{a,b,c} - \mathbf{Z}_{i\sigma}^{a,b,c})\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{ll}^{a,b,c} = Z_{pl}^{a,b,c} - Z_{ql}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\{U + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\mathbf{Z}_{\rho l}^{a,b,c} - \mathbf{Z}_{\sigma l}^{a,b,c})\}$	$Z_{li}^{a,b,c} = -Z_{qi}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\bar{y}_{pq,\rho\sigma}^{a,b,c}(\mathbf{Z}_{\rho i}^{a,b,c} - \mathbf{Z}_{\sigma i}^{a,b,c})$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{il}^{a,b,c} = -Z_{iq}^{a,b,c} + (\mathbf{Z}_{i\rho}^{a,b,c} - \mathbf{Z}_{i\sigma}^{a,b,c})\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{ll}^{a,b,c} = -Z_{ql}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\{U + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\mathbf{Z}_{\rho l}^{a,b,c} - \mathbf{Z}_{\sigma l}^{a,b,c})\}$

Add	*No mutual coupling*	
p–q	*p is not the reference bus*	*p is the reference bus*
Branch	$Z_{qi}^{a,b,c} = Z_{pi}^{a,b,c}$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{iq}^{a,b,c} = Z_{ip}^{a,b,c}$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{qq}^{a,b,c} = Z_{pq}^{a,b,c} + z_{pq,pq}^{a,b,c}$	$Z_{qi}^{a,b,c} = 0$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{iq}^{a,b,c} = 0$ $i = 1, 2, \ldots, m$ $i \neq q$ $Z_{qq}^{a,b,c} = z_{pq,pq}^{a,b,c}$
Link	$Z_{li}^{a,b,c} = Z_{pi}^{a,b,c} - Z_{qi}^{a,b,c}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{il}^{a,b,c} = Z_{ip}^{a,b,c} - Z_{iq}^{a,b,c}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{ll}^{a,b,c} = Z_{pl}^{a,b,c} - Z_{ql}^{a,b,c} + z_{pq,pq}^{a,b,c}$	$Z_{li}^{a,b,c} = -Z_{qi}^{a,b,c}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{il}^{a,b,c} = -Z_{iq}^{a,b,c}$ $i = 1, 2, \ldots, m$ $i \neq l$ $Z_{ll}^{a,b,c} = -Z_{ql}^{a,b,c} + z_{pq,pq}^{a,b,c}$

Modification of the elements for elimination of *l*th node

$$Z_{ij\,(\text{modified})}^{a,b,c} = Z_{ij\,(\text{before elimination})}^{a,b,c} - Z_{il}^{a,b,c}(Z_{ll}^{a,b,c})^{-1}Z_{lj}^{a,b,c} \qquad i,j = 1, 2, \ldots, m$$

From equation (5.7.25),

$$e_l^{a,b,c} = Z_{ll}^{a,b,c} I_l^{a,b,c} \tag{5.7.39}$$

Substituting from equation (5.7.39) into (5.7.38), it follows, since the resultant equation is valid for all values of $I_l^{a,b,c}$, that

$$Z_{ll} = (Z_{pl}^{a,b,c} - Z_{ql}^{a,b,c}) + (y_{pq,pq}^{a,b,c})^{-1}\{U + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\bar{Z}_{\rho l}^{a,b,c} - \bar{Z}_{\sigma l}^{a,b,c})\} \tag{5.7.40}$$

If there is no mutual coupling between the added link and the elements of the partial network, equations (5.7.32), (5.7.35), and (5.7.40) reduce to

$$\begin{aligned} Z_{li}^{a,b,c} &= Z_{pi}^{a,b,c} - Z_{qi}^{a,b,c} \\ Z_{il}^{a,b,c} &= Z_{ip}^{a,b,c} - Z_{iq}^{a,b,c} \\ Z_{ll}^{a,b,c} &= Z_{pl}^{a,b,c} - Z_{ql}^{a,b,c} + z_{pq,pq}^{a,b,c} \end{aligned}$$

If, in addition, p is the reference node,

$$\begin{aligned} Z_{li}^{a,b,c} &= -Z_{qi}^{a,b,c} \\ Z_{il}^{a,b,c} &= -Z_{iq}^{a,b,c} \\ Z_{ll}^{a,b,c} &= -Z_{ql}^{a,b,c} + z_{pq,pq}^{a,b,c} \end{aligned}$$

Furthermore, if the elements are balanced,

$$Z_{li}^{a,b,c} = Z_{il}^{a,b,c}$$

The fictitious node l is eliminated by short circuiting the link voltage source $e_l^{a,b,c}$. From equation (5.7.25),

$$\bar{E}_{BUS}^{a,b,c} = Z_{BUS}^{a,b,c} \bar{I}_{BUS}^{a,b,c} + \bar{Z}_{il}^{a,b,c} I_l^{a,b,c} \tag{5.7.41}$$

and

$$e_l^{a,b,c} = \bar{Z}_{lj}^{a,b,c} \bar{I}_{BUS}^{a,b,c} + Z_{ll}^{a,b,c} I_l^{a,b,c} = 0 \tag{5.7.42}$$

where $i, j = 1, 2, \ldots, m$. Solving for $I_l^{a,b,c}$ from equation (5.7.42) and substituting into (5.7.41),

$$\bar{E}_{BUS}^{a,b,c} = \{Z_{BUS}^{a,b,c} - \bar{Z}_{il}^{a,b,c}(Z_{ll}^{a,b,c})^{-1}\bar{Z}_{lj}^{a,b,c}\}\bar{I}_{BUS}^{a,b,c}$$

Therefore,

$$Z_{ij\,(\text{modified})}^{a,b,c} = Z_{ij\,(\text{before elimination})}^{a,b,c} - Z_{il}^{a,b,c}(Z_{ll}^{a,b,c})^{-1}Z_{lj}^{a,b,c}$$

A summary of equations for the formation of the three-phase bus impedance matrix is given in Table 5.1. These equations can be written in terms of symmetrical or Clarke's components.

5.8 *Modification of the three-phase bus impedance matrix for changes in the network*

The formulas given in Table 5.1 can be used to modify a three-phase bus impedance matrix when an element is added to the network. These formulas can be used also when an element not mutually coupled to other elements of the network is removed or its impedance is changed. The procedures are similar to those used for single-phase networks. When an element is removed, the modified three-phase bus impedance matrix can be obtained by adding a parallel element whose three-phase impedance is equal to the negative of the impedance of the element to be removed. When the impedance of an element is to be changed, the modified three-phase bus impedance matrix can be obtained by adding a parallel element such that the equivalent three-phase impedance of the two elements is the desired value.

The same procedures as those used for single-phase networks can be employed to derive an equation for modifying the submatrices of the three-phase bus impedance matrix when mutually coupled elements are removed or their impedances are changed. This equation is

$$Z'^{a,b,c}_{ij} = Z^{a,b,c}_{ij} + (\bar{Z}^{a,b,c}_{i\alpha} - \bar{Z}^{a,b,c}_{i\beta})[M^{a,b,c}]^{-1}[\Delta y^{a,b,c}_s](\bar{Z}^{a,b,c}_{\gamma j} - \bar{Z}^{a,b,c}_{\delta j})$$

where

$$[\Delta y^{a,b,c}_s] = [y^{a,b,c}_s] - [y'^{a,b,c}_s]$$
$$[M^{a,b,c}] = [U - [\Delta y^{a,b,c}_s]([Z^{a,b,c}_{\gamma\alpha}] - [Z^{a,b,c}_{\delta\alpha}] - [Z^{a,b,c}_{\gamma\beta}] + [Z^{a,b,c}_{\delta\beta}])]$$

5.9 Example of formation and modification of three-phase network matrices

The methods of forming three-phase network matrices by transformation and by algorithm will be illustrated using the sample system shown in Fig. 5.6*a*.

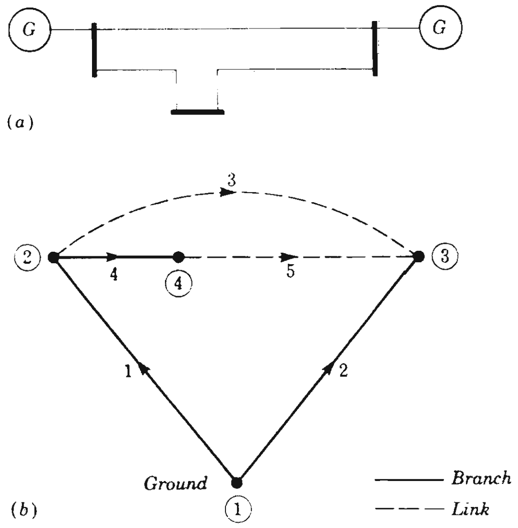

Fig. 5.6 Sample three-phase system. (a) Single line diagram; (b) tree and cotree of oriented connected graph.

Problem

a. Form the bus incidence matrix A with ground as reference.
b. Form the bus admittance matrix Y_{BUS} by transformation.
c. Form the bus impedance matrix Z_{BUS} using the algorithm.
d. Modify the three-phase bus impedance matrix obtained in part *c* to remove element 3 between bus 2 and bus 3.

Table 5.2 Three-phase impedances for sample system

	Self				Mutual			
Element number	*Bus code* p–q	*Impedance* $z^{a,b,c}_{pq,pq}$			*Bus code* r–s	*Impedance* $z^{a,b,c}_{pq,rs}$		
1	1–2	0.080	−0.025	−0.020				
		−0.020	0.080	−0.025				
		−0.025	−0.020	0.080				
2	1–3	0.080	−0.025	−0.020				
		−0.020	0.080	−0.025				
		−0.025	−0.020	0.080				
3	2–3	1.50	0.50	0.50				
		0.50	1.50	0.50				
		0.50	0.50	1.50				
4	2–4	0.60	0.20	0.20	2–3	0.20	0.20	0.20
		0.20	0.60	0.20		0.20	0.20	0.20
		0.20	0.20	0.60		0.20	0.20	0.20
5	4–3	0.90	0.30	0.30	2–3	0.30	0.30	0.30
		0.30	0.90	0.30		0.30	0.30	0.30
		0.30	0.30	0.90		0.30	0.30	0.30

Solution

The data for the sample three-phase system is given in Table 5.2. The impedances for this system are represented by real numbers equal to the generator and line reactances. The branches and links of the oriented connected graph for the single-line representation of the system are shown in Fig. 5.6*b*.

a. The bus incidence matrix is

$$A = \begin{array}{c|ccc} e \diagdown \text{bus} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline 1 & -U & & \\ 2 & & -U & \\ 4 & U & & -U \\ \hline 3 & U & -U & \\ 5 & & -U & U \end{array}$$

where U is a 3 × 3 unit matrix.

b. The primitive impedance matrix of the three-phase system from Table 5.2 is

$$[z^{a,b,c}] = \begin{array}{cc|ccc|ccc|ccc|ccc|ccc}
 & e & & 1 & & & 2 & & & 4 & & & 3 & & & 5 & \\
e & & a & b & c & a & b & c & a & b & c & a & b & c & a & b & c \\
\hline
 & a & .080 & -.025 & -.020 & & & & & & & & & & & & \\
1 & b & -.020 & .080 & -.025 & & & & & & & & & & & & \\
 & c & -.025 & -.020 & .080 & & & & & & & & & & & & \\
\hline
 & a & & & & .080 & -.025 & -.020 & & & & & & & & & \\
2 & b & & & & -.020 & .080 & -.025 & & & & & & & & & \\
 & c & & & & -.025 & -.020 & .080 & & & & & & & & & \\
\hline
 & a & & & & & & & .600 & .200 & .200 & .200 & .200 & 200 & & & \\
4 & b & & & & & & & .200 & .600 & .200 & .200 & .200 & .200 & & & \\
 & c & & & & & & & .200 & .200 & .600 & .200 & .200 & .200 & & & \\
\hline
 & a & & & & & & & .200 & 200 & .200 & 1.500 & .500 & 500 & 300 & 300 & .300 \\
3 & b & & & & & & & .200 & .200 & 200 & .500 & 1.500 & .500 & .300 & 300 & 300 \\
 & c & & & & & & & .200 & .200 & .200 & .500 & .500 & 1.500 & .300 & 300 & 300 \\
\hline
 & a & & & & & & & & & & .300 & 300 & .300 & .900 & 300 & 300 \\
5 & b & & & & & & & & & & .300 & 300 & 300 & .300 & .900 & 300 \\
 & c & & & & & & & & & & .300 & .300 & .300 & .300 & .300 & .900
\end{array}$$

The primitive admittance matrix is the inverse of $[z^{a,b,c}]$.

The bus admittance matrix obtained by singular transformation is

$$Y^{a,b,c}_{BUS} = A^t[y^{a,b,c}]A$$

		1			2			4			3			5	
	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
a	−1						1			1					
① b		−1						1			1				
c			−1						1			1			
a				−1						−1			−1		
= ② b					−1						−1			−1	
c						−1						−1			1
a							−1						1		
③ b								−1						1	
c									−1						1

		1			2			4			3			5	
	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
1 a	16.0163	6.5151	6.0400												
b	6.0400	16.0163	6.5151												
c	6.5151	6.0400	16.0163												
2 a				16.0163	6.5151	6.0400									
b				6.0400	16.0163	6.5151									
c				6.5151	6.0400	16.0163									
4 a							2.0750	.4250	.4250	.1250	.1250	−.1250	.0750	.0750	.0750
b							.4250	2.0750	.4250	.1250	.1250	−.1250	.0750	.0750	.0750
c							.4250	.4250	2.0750	.1250	−.1250	−.1250	.0750	.0750	.0750
3 a							.1250	.1250	.1250	.8750	−.1250	.1250	.1250	−.1250	.1250
b							.1250	.1250	.1250	.1250	.8750	−.1250	−.1250	.1250	.1250
c							.1250	.1250	.1250	.1250	.1250	.8750	.1250	[illegible]	.1250
5 a							.0750	.0750	.0750	.1250	.1250	.1250	1.4083	[illegible]	.5833
b							.0750	.0750	.0750	.1250	−.1250	.1250	.2583	1.4083	.2583
c							.0750	.0750	.0750	.1250	−.1250	−.1250	−.5833	.2583	1.4083

		①			②			③	
	a	b	c	a	b	c	a	b	c
a	−1								
1 b		−1							
c			−1						
a				−1					
2 b					−1				
c						−1			
a	1						−1		
4 b		1						−1	
c			1						−1
a	1			−1					
3 b		1			−1				
c			1			−1			
a				−1			1		
5 b					−1			1	
c						1			1

		①			②			③	
	a	b	c	a	b	c	a	b	c
a	18.7163	5.7151	5.2400	−.7000	.3000	.3000	−2.0000	.5000	.5000
① b	5.2400	18.7163	5.7151	.3000	−.7000	.3000	.5000	−2.0000	.5000
c	5.7151	5.2400	18.7163	.3000	.3000	−.7000	.5000	.5000	−2.0000
a	.7000	.3000	.3000	18.0496	5.8818	5.4067	−1.3333	.3333	.3333
= ② b	.3000	.7000	.3000	5.4067	18.0496	5.8818	.3333	1.3333	.3333
c	.3000	.3000	.7000	5.8818	5.4067	18.0496	.3333	.3333	1.3333
a	−2.0000	.5000	.5000	1.3333	.3333	.3333	3.3333	.8333	.8333
③ b	.5000	−2.0000	.5000	.3333	−1.3333	.3333	−.8333	3.3333	.8333
c	.5000	.5000	−2.0000	.3333	.3333	−1.3333	−.8333	−.8333	3.3333

c. The bus impedance matrix will be formed by first adding all branches and then adding the links.

Step 1. Start with element 1, the branch from $p = 1$ to $q = 2$. The elements of the bus impedance matrix of the partial network are

$$Z^{a,b,c}_{BUS} = \begin{array}{cc|c|c|c|}
 & & & \textcircled{2} & \\
 & & a & b & c \\ \hline
 & a & 0.080 & -0.025 & -0.020 \\ \hline
\textcircled{2} & b & -0.020 & 0.080 & -0.025 \\ \hline
 & c & -0.025 & -0.020 & 0.080 \\ \hline
\end{array}$$

Step 2. Add element 2, the branch from $p = 1$ to $q = 3$. This adds a new bus and the bus impedance matrix is

$$Z^{a,b,c}_{BUS} = \begin{array}{cc|c|c|c|c|c|c|}
 & & & \textcircled{2} & & & \textcircled{3} & \\
 & & a & b & c & a & b & c \\ \hline
 & a & 0.080 & -0.025 & -0.020 & & & \\ \hline
\textcircled{2} & b & -0.020 & 0.080 & -0.025 & & & \\ \hline
 & c & -0.025 & -0.020 & 0.080 & & & \\ \hline
 & a & & & & 0.080 & -0.025 & -0.020 \\ \hline
\textcircled{3} & b & & & & -0.020 & 0.080 & -0.025 \\ \hline
 & c & & & & -0.025 & -0.020 & 0.080 \\ \hline
\end{array}$$

Step 3. Add element 4, the branch from $p = 2$ to $q = 4$. This element is not connected to the reference node and its addition creates a new bus. Using the formulas in Table 5.1

$$Z^{a,b,c}_{qi} = Z^{a,b,c}_{pi} \qquad i = 2, 3$$
$$Z^{a,b,c}_{iq} = Z^{a,b,c}_{ip} \qquad i = 2, 3$$

and

$$Z^{a,b,c}_{qq} = Z^{a,b,c}_{pq} + z^{a,b,c}_{pq,pq}$$

The bus impedance matrix is

$$Z_{BUS}^{a,b,c} = \begin{array}{cc|ccc|ccc|ccc|}
 & & & \textcircled{2} & & & \textcircled{3} & & & \textcircled{4} & \\
 & & a & b & c & a & b & c & a & b & c \\
\hline
 & a & 0.080 & -0.025 & -0.020 & & & & 0.080 & -0.025 & -0.020 \\
\textcircled{2} & b & -0.020 & 0.080 & -0.025 & & & & -0.020 & 0.080 & -0.025 \\
 & c & -0.025 & -0.020 & 0.080 & & & & -0.025 & -0.020 & 0.080 \\
\hline
 & a & & & & 0.080 & -0.025 & -0.020 & & & \\
\textcircled{3} & b & & & & -0.020 & 0.080 & -0.025 & & & \\
 & c & & & & -0.025 & -0.020 & 0.080 & & & \\
\hline
 & a & 0.080 & -0.025 & -0.020 & & & & 0.680 & 0.175 & 0.180 \\
\textcircled{4} & b & -0.020 & 0.080 & -0.025 & & & & 0.180 & 0.680 & 0.175 \\
 & c & -0.025 & -0.020 & 0.080 & & & & 0.175 & 0.180 & 0.680 \\
\hline
\end{array}$$

Step 4. Add element 5, the link from $p = 4$ to $q = 3$. This element is not connected to the reference node and is not mutually coupled to an existing element of the partial network. The elements of the rows and columns corresponding to the fictitious node l are obtained from

$$Z_{li}^{a,b,c} = Z_{pi}^{a,b,c} - Z_{qi}^{a,b,c} \qquad i = 2, 3, 4$$
$$Z_{il}^{a,b,c} = Z_{ip}^{a,b,c} - Z_{iq}^{a,b,c} \qquad i = 2, 3, 4$$

and

$$Z_{ll}^{a,b,c} = Z_{pl}^{a,b,c} - Z_{ql}^{a,b,c} + z_{pq,pq}^{a,b,c}$$

Thus, the augmented impedance matrix is

		②			③			④			*l*		
		a	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*
	a	.080	− .025	− .020				.080	− .025	− .020	.080	− .025	− .020
②	*b*	− .020	.080	− .025				− .020	.080	− .025	− .020	.080	− .025
	c	− .025	− .020	.080				− .025	− .020	.080	− .025	− .020	.080
	a				.080	− .025	− .020				− .080	.025	.020
③	*b*				− .020	.080	− .025				.020	− .080	.025
	c				− .025	− .020	.080				.025	.020	− .080
	a	.080	− .025	− .020				.680	.175	.180	.680	.175	.180
④	*b*	− .020	.080	− .025				.180	.680	.175	.180	.680	.175
	c	− .025	− .020	.080				.175	.180	.680	.175	.180	.680
	a	.080	− .025	− .020	− .080	.025	.020	.680	.175	.180	1.660	.450	.460
l	*b*	− .020	.080	− .025	.020	− .080	.025	.180	.680	.175	.460	1.660	.450
	c	− .025	− .020	.080	.025	.020	− .080	.175	.180	.680	.450	.460	1.660

The rows and columns corresponding to the fictitious node l are eliminated using the formula

$$Z^{a,b,c}_{ij\,(\text{modified})} = Z^{a,b,c}_{ij\,(\text{before elimination})} - Z^{a,b,c}_{il}(Z^{a,b,c}_{ll})^{-1}Z^{a,b,c}_{lj}$$

Then, the bus impedance matrix is

$$Z^{a,b,c}_{BUS} = \begin{array}{cc|ccc|ccc|ccc|}
 & & & ② & & & ③ & & & ④ & \\
 & & a & b & c & a & b & c & a & b & c \\
\hline
 & a & .0740 & -.0219 & -.0176 & .0060 & -.0031 & -.0024 & .0468 & -.0144 & -.0115 \\
② & b & -.0176 & .0740 & -.0219 & -.0024 & .0060 & -.0031 & -.0115 & .0468 & -.0144 \\
 & c & -.0219 & -.0176 & .0740 & -.0031 & -.0024 & .0060 & -.0144 & -.0115 & .0468 \\
\hline
 & a & .0060 & -.0031 & -.0024 & .0740 & -.0219 & -.0176 & .0332 & -.0106 & -.0085 \\
③ & b & -.0024 & .0060 & -.0031 & -.0176 & .0740 & -.0219 & -.0085 & .0332 & -.0106 \\
 & c & -.0031 & -.0024 & .0060 & -.0219 & -.0176 & .0740 & -.0106 & -.0085 & .0032 \\
\hline
 & a & .0468 & -.0144 & -.0115 & .0332 & -.0106 & -.0085 & 4014 & .1071 & .1097 \\
④ & b & -.0115 & .0468 & -.0144 & -.0085 & .0332 & -.0106 & .1097 & 4014 & .1071 \\
 & c & -.0144 & -.0115 & .0468 & -.0106 & -.0085 & .0332 & .1071 & .1097 & .4014 \\
\hline
\end{array}$$

Step 5. Add element 3, the link from $p = 2$ to $q = 3$ mutually coupled to elements 4 and 5. This element is not connected to the reference node. The elements of the rows and columns corresponding to the fictitious node l are obtained from

$$
\begin{aligned}
Z_{li}^{a,b,c} &= Z_{pi}^{a,b,c} - Z_{qi}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\bar{y}_{pq,\rho\sigma}^{a,b,c}(\bar{Z}_{\rho i}^{a,b,c} - \bar{Z}_{\sigma i}^{a,b,c}) && i = 2, 3, 4 \\
Z_{il}^{a,b,c} &= Z_{ip}^{a,b,c} - Z_{iq}^{a,b,c} + (\bar{Z}_{i\rho}^{a,b,c} - \bar{Z}_{i\sigma}^{a,b,c})\bar{y}_{\rho\sigma,pq}^{a,b,c}(y_{pq,pq}^{a,b,c})^{-1} && i = 2, 3, 4
\end{aligned}
$$

and

$$
Z_{ll}^{a,b,c} = Z_{pl}^{a,b,c} - Z_{ql}^{a,b,c} + (y_{pq,pq}^{a,b,c})^{-1}\{U + \bar{y}_{pq,\rho\sigma}^{a,b,c}(\bar{Z}_{\rho l}^{a,b,c} - \bar{Z}_{\sigma l}^{a,b,c})\}
$$

The submatrix $Z_{12}^{a,b,c}$ is

$$Z_{12}^{a,b,c} = Z_{22}^{a,b,c} - Z_{32}^{a,b,c} + (y_{23,23}^{a,b,c})^{-1} \left[\begin{array}{c|c} y_{23,24}^{a,b,c} & y_{23,43}^{a,b,c} \end{array}\right] \left[\begin{array}{c} Z_{22}^{a,b,c} - Z_{42}^{a,b,c} \\ \hline Z_{42}^{a,b,c} - Z_{32}^{a,b,c} \end{array}\right]$$

$$= \left[\begin{array}{ccc} .0740 & -.0219 & -.0176 \\ -.0176 & .0740 & -.0219 \\ -.0219 & -.0176 & .0740 \end{array}\right] - \left[\begin{array}{ccc} .0060 & -.0031 & -.0024 \\ -.0024 & .0060 & -.0031 \\ -.0031 & -.0024 & .0060 \end{array}\right]$$

$$+ \left[\begin{array}{ccc} 1.2 & .2 & .2 \\ .2 & 1.2 & .2 \\ .2 & .2 & 1.2 \end{array}\right] \left[\begin{array}{ccc|ccc} -.1250 & -.1250 & -.1250 & -.1250 & -.1250 & -.1250 \\ -.1250 & -.1250 & -.1250 & -.1250 & -.1250 & .1250 \\ -.1250 & -.1250 & -.1250 & -.1250 & -.1250 & -.1250 \end{array}\right] \left[\begin{array}{ccc} .0272 & -.0075 & -.0061 \\ -.0061 & .0272 & -.0075 \\ -.0075 & -.0061 & .0272 \\ \hline .0408 & -.0113 & -.0091 \\ -.0091 & .0408 & -.0113 \\ -.0113 & -.0091 & .0408 \end{array}\right]$$

$$= \left[\begin{array}{ccc} .0612 & -.0256 & -.0220 \\ -.0220 & .0612 & -.0256 \\ -.0256 & -.0220 & .0612 \end{array}\right]$$

The submatrix $Z_{2l}^{a,b,c}$ is

$$Z_{2l}^{a,b,c} = Z_{22}^{a,b,c} - Z_{23}^{a,b,c} + \begin{bmatrix} Z_{22}^{a,b,c} - Z_{24}^{a,b,c} & Z_{24}^{a,b,c} - Z_{23}^{a,b,c} \end{bmatrix} \begin{bmatrix} y_{24,23}^{a,b,c} \\ y_{43,23}^{a,b,c} \end{bmatrix} (y_{23,23}^{a,b,c})^{-1}$$

$$= \begin{bmatrix} .0740 & -.0219 & -.0176 \\ -.0176 & .0740 & -.0219 \\ -.0219 & -.0176 & .0740 \end{bmatrix} - \begin{bmatrix} .0060 & -.0031 & -.0024 \\ -.0024 & .0060 & -.0031 \\ -.0031 & -.0024 & .0060 \end{bmatrix}$$

$$+ \left[\begin{array}{ccc|ccc} .0272 & -.0075 & -.0061 & .0408 & -.0113 & -.0091 \\ -.0061 & .0272 & -.0075 & -.0091 & .0408 & -.0113 \\ -.0075 & -.0061 & .0272 & -.0113 & -.0091 & .0408 \end{array}\right] \left[\begin{array}{ccc} -.1250 & -.1250 & -.1250 \\ -.1250 & -.1250 & -.1250 \\ -.1250 & -.1250 & -.1250 \\ \hline -.1250 & -.1250 & -.1250 \\ -.1250 & .1250 & .1250 \\ -.1250 & -.1250 & -.1250 \end{array}\right] \begin{bmatrix} 1.2 & .2 & .2 \\ .2 & 1.2 & .2 \\ .2 & .2 & 1.2 \end{bmatrix}$$

$$= \begin{bmatrix} .0612 & -.0256 & -.0220 \\ -.0220 & .0612 & -.0256 \\ -.0256 & -.0220 & .0612 \end{bmatrix}$$

In a similar manner the remaining submatrices $Z_{li}^{a,b,c}$ and $Z_{il}^{a,b,c}$ for $i = 3, 4$ can be calculated. In this problem

$Z_{2l}^{a,b,c} = Z_{l2}^{a,b,c}$ $\quad Z_{3l}^{a,b,c} = Z_{l3}^{a,b,c}$ $\quad$ and $\quad Z_{4l}^{a,b,c} = Z_{l4}^{a,b,c}$

since the elements of the network are balanced.

The submatrix $Z_{ll}^{a,b,c}$ is

$$Z_{ll}^{a,b,c} = Z_{2l}^{a,b,c} - Z_{3l}^{a,b,c} + \left(y_{22,22}^{a,b,c}\right)^{-1}\left\{U + \begin{vmatrix} y_{22,34}^{a,b,c} & y_{22,43}^{a,b,c} \end{vmatrix} \begin{vmatrix} Z_{3l}^{a,b,c} - Z_{4l}^{a,b,c} \\ Z_{4l}^{a,b,c} - Z_{3l}^{a,b,c} \end{vmatrix}\right\}$$

$$= \begin{vmatrix} .0612 & -.0256 & -.0220 \\ -.0220 & .0612 & -.0256 \\ -.0256 & -.0220 & .0612 \end{vmatrix} - \begin{vmatrix} -.0612 & .0256 & .0220 \\ .0220 & -.0612 & .0256 \\ .0256 & .0220 & -.0612 \end{vmatrix}$$

$$+ \begin{vmatrix} 1.2 & .2 & .2 \\ .2 & 1.2 & .2 \\ .2 & .2 & 1.2 \end{vmatrix} \left(\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + \begin{vmatrix} -.1250 & -.1250 & -.1250 & -.1250 & -.1250 & -.1250 \\ -.1250 & -.1250 & -.1250 & -.1250 & -.1250 & -.1250 \\ -.1250 & .1250 & -.1250 & -.1250 & -.1250 & -.1250 \end{vmatrix} \begin{vmatrix} .0490 & .0204 & -.0176 \\ -.0176 & .0490 & -.0204 \\ -.0204 & -.0176 & .0490 \\ \hline .0734 & -.0308 & -.0264 \\ -.0264 & .0734 & -.0308 \\ -.0308 & -.0264 & .0734 \end{vmatrix} \right)$$

$$= \begin{vmatrix} 1.3170 & .1434 & .1506 \\ .1506 & 1.3170 & .1434 \\ .1434 & .1506 & 1.3170 \end{vmatrix}$$

The augmented impedance matrix is

		②			③			④			l		
		a	b	c	a	b	c	a	b	c	a	b	c
	a	.0740	— .0219	— .0176	.0060	— .0031	— .0024	.0468	— .0144	— .0115	.0612	— .0256	— .0220
②	b	— .0176	.0740	— .0219	— .0024	.0060	— .0031	— .0115	.0468	— .0144	— .0220	.0612	— .0256
	c	— .0219	— .0176	.0740	— .0031	— .0024	.0060	— .0144	— .0115	.0468	— .0256	— .0220	.0612
	a	.0060	— .0031	— .0024	.0740	— .0219	— .0176	.0332	— .0106	— .0085	— .0612	.0256	.0220
③	b	— .0024	.0060	— .0031	— .0176	.0740	— .0219	— .0085	.0332	— .0106	.0220	— .0612	.0256
	c	— .0031	— .0024	.0060	— .0219	— .0176	.0740	— .0106	— .0085	.0332	.0256	.0220	— .0612
	a	.0468	— .0144	— .0115	.0332	— .0106	— .0085	.4014	.1071	.1097	.0122	— .0052	— .0044
④	b	— .0115	.0468	— .0144	— .0085	.0332	— .0106	.1097	.4014	.1071	— .0044	.0122	— .0052
	c	— .0144	— .0115	.0468	— .0106	— .0085	.0332	.1071	.1097	.4014	— .0052	— .0044	.0122
	a	.0612	— .0256	— .0220	— .0612	.0256	.0220	.0122	— .0052	— .0044	1.3170	.1434	.1506
l	b	— .0220	.0612	— .0256	.0220	— .0612	.0256	— .0044	.0122	— .0052	.1506	1.3170	.1434
	c	— .0256	— .0220	.0612	.0256	.0220	— .0612	— .0052	— .0044	.0122	.1434	.1506	1.3170

Eliminating the rows and columns corresponding to the fictitious node l, the bus impedance matrix of the system is

$$Z_{BUS}^{a,b,c} = \begin{array}{cc|ccc|ccc|ccc}
 & & & (2) & & & (3) & & & (4) & \\
 & & a & b & c & a & b & c & a & b & c \\
\hline
 & a & .0699 & -.0196 & -.0159 & .0101 & -.0054 & -.0041 & .0460 & -.0139 & -.0112 \\
② & b & -.0159 & .0699 & -.0196 & -.0041 & .0101 & -.0054 & -.0112 & .0460 & -.0139 \\
 & c & -.0196 & -.0159 & .0699 & -.0054 & -.0041 & .0101 & -.0139 & -.0112 & .0460 \\
\hline
 & a & .0101 & -.0054 & -.0041 & .0699 & -.0196 & -.0159 & .0340 & -.0111 & -.0088 \\
③ & b & -.0041 & .0101 & -.0054 & -.0159 & .0699 & -.0196 & -.0088 & .0340 & -.0111 \\
 & c & -.0054 & -.0041 & .0101 & -.0196 & -.0159 & .0699 & -.0111 & -.0088 & .0340 \\
\hline
 & a & .0460 & -.0139 & -.0112 & .0340 & -.0111 & -.0088 & .4012 & .1072 & .1098 \\
④ & b & -.0112 & .0460 & -.0139 & -.0088 & .0340 & -.0111 & .1098 & .4012 & .1072 \\
 & c & -.0139 & -.0112 & .0460 & -.0111 & -.0088 & .0340 & .1072 & .1098 & .4012
\end{array}$$

This matrix can be checked by inverting the bus admittance matrix obtained in part b of this problem.

d. The modified elements of the bus impedance matrix for the removal of the network element 2–3 mutually coupled to the network elements 2–4 and 4–3 are obtained from

$$Z'^{a,b,c}_{ij} = Z^{a,b,c}_{ij} + (\bar{Z}^{a,b,c}_{i\alpha} - \bar{Z}^{a,b,c}_{i\beta})[M^{a,b,c}]^{-1}[\Delta y^{a,b,c}_{s}](\bar{Z}^{a,b,c}_{\gamma j} - \bar{Z}^{a,b,c}_{\delta j}) \qquad \begin{aligned} i, j &= 2, 3, 4 \\ \alpha, \gamma &= 2, 4, 2 \\ \beta, \delta &= 4, 3, 3 \end{aligned}$$

The original primitive admittance submatrix is

		2–4			4–3			2–3	
	a	b	c	a	b	c	a	b	c
a	2.0750	−.4250	−.4250	.0750	.0750	.0750	−.1250	−.1250	−.1250
2–4 b	−.4250	2.0750	−.4250	.0750	.0750	.0750	−.1250	−.1250	−.1250
c	−.4250	−.4250	2.0750	.0750	.0750	.0750	−.1250	−.1250	−.1250
a	.0750	.0750	.0750	1.4083	−.2583	−.2583	−.1250	−.1250	−.1250
$[y^{a,b,c}_{s}]$ = 4–3 b	.0750	.0750	.0750	−.2583	1.4083	−.2583	−.1250	−.1250	−.1250
c	.0750	.0750	.0750	−.2583	−.2583	1.4083	−.1250	−.1250	−.1250
a	−.1250	−.1250	−.1250	−.1250	−.1250	−.1250	.8750	−.1250	−.1250
2–3 b	−.1250	−.1250	−.1250	−.1250	−.1250	−.1250	−.1250	.8750	−.1250
c	−.1250	−.1250	−.1250	−.1250	−.1250	−.1250	−.1250	−.1250	.8750

The new primitive admittance submatrix is

$$
[y_{s}'^{a,b,c}] = \begin{array}{cc|ccc|ccc|ccc|}
 & & & 2\text{–}4 & & & 4\text{–}3 & & & 2\text{–}3 & \\
 & & a & b & c & a & b & c & a & b & c \\
\hline
 & a & 2.0000 & -.5000 & -.5000 & & & & & & \\
2\text{–}4 & b & -.5000 & 2.0000 & -.5000 & & & & & & \\
 & c & -.5000 & -.5000 & 2.0000 & & & & & & \\
\hline
 & a & & & & 1.3333 & -.3333 & -.3333 & & & \\
4\text{–}3 & b & & & & -.3333 & 1.3333 & -.3333 & & & \\
 & c & & & & -.3333 & -.3333 & 1.3333 & & & \\
\hline
 & a & & & & & & & & & \\
2\text{–}3 & b & & & & & & & & & \\
 & c & & & & & & & & & \\
\hline
\end{array}
$$

Then, $[\Delta y_s^{a,b,c}] = [y_s^{a,b,c}] - [y_s'^{a,b,c}]$

		2–4			4–3			2–3		
		a	b	c	a	b	c	a	b	c
	a	.0750	.0750	.0750	.0750	.0750	.0750	— .1250	— .1250	— .1250
2–4	b	.0750	.0750	.0750	.0750	.0750	.0750	— .1250	— .1250	— .1250
	c	.0750	.0750	.0750	.0750	.0750	.0750	— .1250	— .1250	— .1250
	a	.0750	.0750	.0750	.0750	.0750	.0750	— .1250	— .1250	— .1250
$[\Delta y_s^{a,b,c}] =$ 4–3	b	.0750	.0750	.0750	.0750	.0750	.0750	— .1250	— .1250	— .1250
	c	.0750	.0750	.0750	.0750	.0750	.0750	— .1250	— .1250	— .1250
	a	— .1250	— .1250	— .1250	— .1250	— .1250	— .1250	.8750	— .1250	— .1250
2–3	b	— .1250	— .1250	— .1250	— .1250	— .1250	— .1250	— .1250	.8750	— .1250
	c	— .1250	— .1250	— .1250	— .1250	— .1250	— .1250	— .1250	— .1250	.8750

Also

$$[M^{a,b,c}] = \{U - [\Delta y_s^{a,b,c}]([Z_{\gamma\alpha}^{a,b,c}] - [Z_{\delta\alpha}^{a,b,c}] - [Z_{\gamma\beta}^{a,b,c}] + [Z_{\delta\beta}^{a,b,c}])\}$$

$$\alpha, \gamma = 2, 4, 2$$

$$\beta, \delta = 4, 3, 3$$

where $[Z_{\gamma\alpha}^{a,b,c}] = \begin{array}{|c|c|c|} \hline Z_{22}^{a,b,c} & Z_{24}^{a,b,c} & Z_{22}^{a,b,c} \\ \hline Z_{42}^{a,b,c} & Z_{44}^{a,b,c} & Z_{42}^{a,b,c} \\ \hline Z_{22}^{a,b,c} & Z_{24}^{a,b,c} & Z_{22}^{a,b,c} \\ \hline \end{array}$

$$[Z_{\delta\alpha}^{a,b,c}] = \begin{array}{|c|c|c|} \hline Z_{42}^{a,b,c} & Z_{44}^{a,b,c} & Z_{42}^{a,b,c} \\ \hline Z_{32}^{a,b,c} & Z_{34}^{a,b,c} & Z_{32}^{a,b,c} \\ \hline Z_{32}^{a,b,c} & Z_{34}^{a,b,c} & Z_{32}^{a,b,c} \\ \hline \end{array}$$

$$[Z_{\gamma\beta}^{a,b,c}] = \begin{array}{|c|c|c|} \hline Z_{24}^{a,b,c} & Z_{23}^{a,b,c} & Z_{23}^{a,b,c} \\ \hline Z_{44}^{a,b,c} & Z_{43}^{a,b,c} & Z_{43}^{a,b,c} \\ \hline Z_{24}^{a,b,c} & Z_{23}^{a,b,c} & Z_{23}^{a,b,c} \\ \hline \end{array}$$

$$[Z_{\delta\beta}^{a,b,c}] = \begin{array}{|c|c|c|} \hline Z_{44}^{a,b,c} & Z_{43}^{a,b,c} & Z_{43}^{a,b,c} \\ \hline Z_{34}^{a,b,c} & Z_{33}^{a,b,c} & Z_{33}^{a,b,c} \\ \hline Z_{34}^{a,b,c} & Z_{33}^{a,b,c} & Z_{33}^{a,b,c} \\ \hline \end{array}$$

and $Z_{22}^{a,b,c}$, $Z_{24}^{a,b,c}$, and so forth, are obtained from the original bus impedance matrix.

Substituting, then

$$[M^{a,b,c}] = \left[\begin{array}{ccc|ccc|ccc}
1.00135 & .00135 & .00135 & .00203 & .00203 & .00203 & .00338 & .00338 & .00338 \\
.00135 & 1.00135 & .00135 & .00203 & .00203 & .00203 & .00338 & .00338 & .00338 \\
.00135 & .00135 & 1.00135 & .00203 & .00203 & .00203 & .00338 & .00338 & .00338 \\
\hline
.00135 & .00135 & .00135 & 1.00203 & .00203 & .00203 & .00338 & .00338 & .00338 \\
.00135 & .00135 & .00135 & .00203 & 1.00203 & .00203 & .00338 & .00338 & .00338 \\
.00135 & .00135 & .00135 & .00203 & .00203 & 1.00203 & .00338 & .00338 & .00338 \\
\hline
-.04105 & .01815 & .01615 & -.06165 & .02715 & .02435 & .89730 & .04530 & .04050 \\
.01615 & -.04105 & .01815 & .02435 & -.06165 & .02715 & .04050 & .89730 & .04530 \\
.01815 & .01615 & -.04105 & .02715 & .02435 & -.06165 & .04530 & .04050 & .89730
\end{array}\right]$$

For the calculation of $Z'^{a,b,c}_{23}$, $i = 2$, $j = 3$, α, $\gamma = 2, 4, 2$. and β, $\delta = 4, 3, 3$. Then

$$(Z^{a,b,c}_{i\alpha} - Z^{a,b,c}_{i\beta}) = \left[\begin{array}{c|c|c} Z^{a,b,c}_{22} - Z^{a,b,c}_{24} & Z^{a,b,c}_{24} - Z^{a,b,c}_{23} & Z^{a,b,c}_{22} - Z^{a,b,c}_{23} \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc|ccc} .0239 & -.0057 & -.0047 & .0359 & -.0085 & -.0071 & 0598 & -\ 0142 & -\ 0118 \\ -.0047 & .0239 & -.0057 & -.0071 & .0359 & -.0085 & -.0118 & .0598 & -.0142 \\ -.0057 & -.0047 & .0239 & -.0085 & -.0071 & .0359 & -.0142 & -.0118 & .0598 \end{array}\right]$$

and

$$(Z^{a,b,c}_{\gamma j} - Z^{a,b,c}_{\delta j}) = \left[\begin{array}{c} Z^{a,b,c}_{23} - Z^{a,b,c}_{43} \\ \hline Z^{a,b,c}_{43} - Z^{a,b,c}_{33} \\ \hline Z^{a,b,c}_{23} - Z^{a,b,c}_{33} \end{array}\right]$$

$$= \left[\begin{array}{ccc} -.0239 & .0057 & .0047 \\ .0047 & -.0239 & .0057 \\ .0057 & .0047 & -.0239 \\ \hline -.0359 & .0085 & .0071 \\ .0071 & -.0359 & .0085 \\ .0085 & .0071 & -.0359 \\ \hline -.0598 & .0142 & .0118 \\ .0118 & -.0598 & .0142 \\ .0142 & .0118 & -.0598 \end{array}\right]$$

$$Z'^{a,b,c}_{23} = Z^{a,b,c}_{23} + \begin{bmatrix} Z^{a,b,c}_{22} - Z^{a,b,c}_{24} & Z^{a,b,c}_{24} - Z^{a,b,c}_{23} & Z^{a,b,c}_{22} - Z^{a,b,c}_{23} \end{bmatrix} [M^{a,b,c}]^{-1}[\Delta y^{a,b,c}_{s}] \begin{bmatrix} Z^{a,b,c}_{23} - Z^{a,b,c}_{43} \\ Z^{a,b,c}_{43} - Z^{a,b,c}_{33} \\ Z^{a,b,c}_{23} - Z^{a,b,c}_{33} \end{bmatrix}$$

$$= \begin{bmatrix} .0101 & -.0054 & -.0041 \\ -.0041 & .0101 & -.0054 \\ -.0054 & -.0041 & .0101 \end{bmatrix} + \begin{bmatrix} -.0042 & .0023 & .0018 \\ .0018 & -.0042 & .0023 \\ .0023 & .0018 & -.0042 \end{bmatrix}$$

$$= \begin{array}{c|ccc|} & a & b & c \\ \hline a & .0059 & -.0031 & -.0023 \\ b & -.0023 & .0059 & -.0031 \\ c & -.0031 & -.0023 & .0059 \\ \hline \end{array}$$

The remaining submatrices of $Z'^{a,b,c}_{BUS}$ are calculated in a similar manner to obtain the final modified matrix.

$$Z'^{a,b,c}_{BUS} = \begin{array}{cc|ccc|ccc|ccc|}
 & & & \textcircled{2} & & & \textcircled{3} & & & \textcircled{4} & \\
 & & a & b & c & a & b & c & a & b & c \\
\hline
 & a & .0741 & -.0218 & -.0177 & .0059 & -.0031 & -.0023 & .0468 & -.0143 & -.0116 \\
\textcircled{2} & b & -.0177 & .0741 & -.0218 & -.0023 & .0059 & -.0031 & -.0116 & .0468 & -.0143 \\
 & c & -.0218 & -.0177 & .0741 & -.0031 & -.0023 & .0059 & -.0143 & -.0116 & .0468 \\
\hline
 & a & .0059 & -.0031 & -.0023 & .0741 & -.0218 & -.0177 & .0332 & -.0107 & -.0084 \\
\textcircled{3} & b & -.0023 & .0059 & -.0031 & -.0177 & .0741 & -.0218 & -.0084 & .0332 & -.0107 \\
 & c & -.0031 & -.0023 & .0059 & -.0218 & -.0177 & .0741 & -.0107 & -.0084 & .0332 \\
\hline
 & a & .0468 & -.0143 & -.0116 & .0332 & -.0107 & -.0084 & 4014 & .1071 & 1097 \\
\textcircled{4} & b & -.0116 & .0468 & -.0143 & -.0084 & .0332 & -.0107 & .1097 & .4014 & .1071 \\
 & c & -.0143 & -.0116 & .0468 & -.0107 & -.0084 & .0332 & 1071 & 1097 & .4014 \\
\hline
\end{array}$$

This matrix checks with that obtained after step 4 in part c except for a slight difference due to round-off error.

Problems

5.1 In Prob. 3.2, the positive and zero sequence data for the sample system shown in Fig. 3.14 is given in Table 3.5. For this system:

a. With ground as reference, form the three-phase incidence matrices A, K, B, $\hat{B}$, C, and $\hat{C}$ for the oriented connected graph selected for Prob. 3.2 and verify the relations:

i. $A_b K^t = U$

ii. $B_l = A_l K^t$

iii. $C_b = -B_l^t$

iv. $\hat{C}\hat{B}^t = U$

b. Neglecting resistance and assuming all negative sequence reactances are equal to the corresponding positive sequence reactances, form the three-phase network matrices $Y_{BUS}^{a,b,c}$ and $Z_{LOOP}^{a,b,c}$ by singular transformations.

c. Neglecting resistance and assuming the positive and negative sequence impedances are equal, form the three-phase network matrix $Z_{BUS}^{a,b,c}$ using the algorithm and ground as reference.

d. Transform $Z_{BUS}^{a,b,c}$ calculated in part *c* to $Z_{BUS}^{0,1,2}$. The submatrices for positive and zero sequences can be verified with those obtained in Prob. 3.2.

5.2 The sequence impedance data for the sample system shown in Fig. 5.7 is given in Table 5.3. Selecting ground as reference (node 0), compute $Z_{BUS}^{0,1,2}$ using the algorithm.

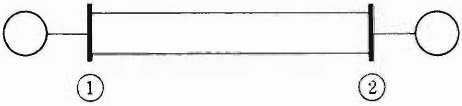

Fig. 5.7 Sample system for Prob. 5.2.

Table 5.3 Sequence impedance data of sample power system for Prob. 5.2

	Self				*Mutual*			
Element number	*Bus code* p–q	*Impedance* $z^{0,1,2}_{pq,pq}$			*Bus code* r–s	*Impedance* $z^{0,1,2}_{pq,rs}$		
1	0–1	$0 + j0.04$						
			$0 + j0.10$					
				$0 + j0.10$				
2	0–2	$0 + j0.04$						
			$0 + j0.10$					
				$0 + j0.10$				
3	1–2(1)	$0.35 + j0.65$	$0.02 - j0.01$	$-0.02 - j0.01$				
		$-0.02 - j0.01$	$0.15 + j0.60$	$-0.04 + j0.02$				
		$0.02 - j0.01$	$0.04 + j0.02$	$0.15 + j0.60$				
4	1–2(2)	$0.35 + j0.65$	$0.02 - j0.01$	$-0.02 - j0.01$	1–2(1)	$0.20 + j1.20$		
		$-0.02 - j0.01$	$0.15 + j0.60$	$-0.04 + j0.02$				
		$0.02 - j0.01$	$0.04 + j0.02$	$0.15 + j0.60$				

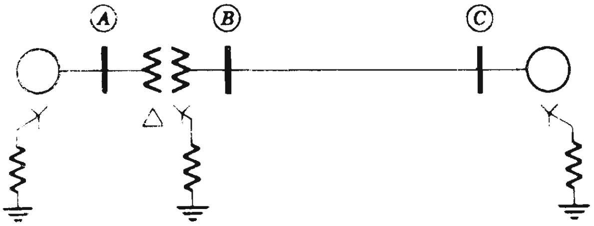

Fig. 5.8 Sample system for Prob. 5.3.

5.3 The reactance data for the three-phase system shown in Fig. 5.8 is Generators A and C:

$$x^{(1)} = x^{(2)} = 0.1$$
$$x^{(0)} = 0.04$$
$$x_g = 0.02$$

Transformer A–B:

$$x^{(1)} = x^{(2)} = x^{(0)} = 0.1$$
$$x_g = 0.05$$

Transmission line B–C:

$$x^{a,b,c} = \begin{array}{|c|c|c|} \hline 0.3 & 0.2 & 0 \\ \hline 0.2 & 0.4 & 0.1 \\ \hline 0 & 0.1 & 0.2 \\ \hline \end{array}$$

a. With ground as reference form $Y^{0,1,2}_{BUS}$.
b. Form $Z^{0,1,2}_{BUS}$ using the algorithm.
c. Determine $Z^{a,b,c}_{BUS}$ from $Z^{0,1,2}_{BUS}$ obtained in part b.

5.4 Assume that the transmission line B–C of Prob. 5.3 is balanced and its reactance is

$$x^{a,b,c} = \begin{array}{|c|c|c|} \hline 0.3 & 0.1 & 0.1 \\ \hline 0.1 & 0.3 & 0.1 \\ \hline 0.1 & 0.1 & 0.3 \\ \hline \end{array}$$

a. Compute $Z_{BUS}^{(0)}$, $Z_{BUS}^{(1)}$, and $Z_{BUS}^{(2)}$.

b. Determine $Z_{BUS}^{a,b,c}$ from $Z_{BUS}^{0,1,2}$ obtained in part *a* and compare the the results to those obtained for the unbalanced line in Prob. 5.3, part *c*.

5.5 The sequence impedance data for the sample system shown in Fig. 5.9 is given in Table 5.4. The mutual impedances $z_{12,13}^{0,1,2}$ and $z_{13,12}^{0,1,2}$ are not equal because of the circuit arrangement. For this system compute $Z_{BUS}^{0,1,2}$ using the ground (node 0) as reference.

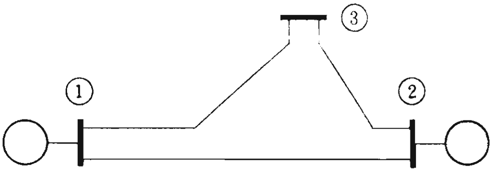

***Fig. 5.9** Sample system for Prob. 5.5.*

Table 5.4 Sequence impedance data of sample power system for Prob. 5.5

		Self			Mutual			
Element Number	Bus code p–q	Impedance $z^{0,1,2}_{pq,pq}$			Bus code r–s	Impedance $z^{0,1,2}_{pq,rs}$		
		0 + j.0400						
1	0–1		0 + j.1000					
				0 + j.1000				
		0 + j.0400						
2	0–2		0 + j.1000					
				0 + j.1000				
		.3504 + j.9965	.0041 − j.0125	−.0065 − j.0022		.1317 + j.1545	−.0166 − j.0117	.0096 − j.0141
3	1–2	−.0065 − j.0022	.0644 + j.4510	−.0244 − j.0152	1–3	−.0018 + j.0202	−.0029 − j.0052	.0050 − j.0028
		.0041 − j.0125	.0257 + j.0136	.0644 + j.4510		0074 + j.0153	−.0049 − j.0029	.0026 − j.0045
		.3504 + j.9965	.0052 − j.0043	−.0129 + j.0027		.1317 + j.1545	.0074 + j.0153	−.0018 + j.0202
4	1–3	−.0129 + j.0027	.0644 + j.4510	−.0247 + j.0156	1–2	.0096 − j.0141	.0026 − j.0045	.0050 − j.0028
		.0052 − j.0043	.0254 + j.0135	.0644 + j.4510		−.0166 − j.0117	.0049 − j.0029	−.0029 − j.0052
		.1776 + j.4936	.0018 − j.0044	−.0047 + j.0006				
5	2–3	−.0047 + j.0006	.0319 + j.2256	−.0124 + j.0076				
		.0018 − j.0044	.0128 + j.0069	.0319 + j.2256				

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chapter 6
Short circuit studies

6.1 Introduction

Short circuit calculations provide currents and voltages on a power system during fault conditions. This information is required to design an adequate protective relaying system and to determine interrupting requirements for circuit breakers at each switching location. Relaying systems must recognize the existence of a fault and initiate circuit breaker operation to disconnect faulted facilities. This action is required to assure minimum disruption of electrical service and to limit damage in the faulted equipment. The currents and voltages resulting from various types of faults occurring at many locations throughout the power system must be calculated to provide sufficient data to develop an effective relaying and switching system. To obtain the required information a special purpose analog computer, called a network analyzer, was used extensively for short circuit studies before digital techniques were available.

The bus frame of reference in admittance form was employed in the first application of digital computers to short circuit studies. This method, which was patterned after similar techniques employed for load flow calculations, used an iterative technique (Coombe and Lewis, 1956). This required a complete iterative solution for each fault type and location. The procedure was time-consuming, particularly if, as was usually the case, the currents and voltages were required for a large number of fault locations. Consequently, this method was not adopted generally.

The development of techniques for applying a digital computer to form the bus impedance matrix made it feasible to use Thevenin's theorem for short circuit calculations. This approach provided an efficient means of determining short circuit currents and voltages because these values can be obtained with few arithmetic operations involving only related portions of the bus impedance matrix.

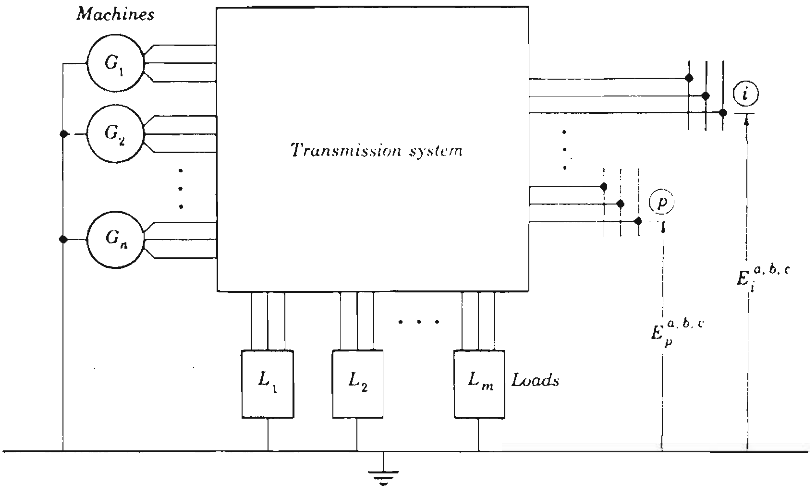


Fig. 6.1 Three-phase representation of a power system.

6.2 Short circuit calculations using Z_{BUS}

System representation

The three-phase representation of a power system under steady state condition is shown in Fig. 6.1. In general, sufficient accuracy in short circuit studies can be obtained with a simplified representation. The simplified three-phase representation is shown in Fig. 6.2 and is obtained by:

1. Representing each machine by a constant voltage behind the machine reactance, transient or subtransient
2. Neglecting shunt connections, e.g., loads, line charging, etc.
3. Setting all transformers at nominal taps

In many short circuit studies, particularly for high voltage systems, it is sufficient to represent transmission line and transformer impedances as real numbers equal to the corresponding reactances.

Fault currents and voltages

The use of the bus impedance matrix provides a convenient means of calculating short circuit currents and voltages when the ground is selected as reference. One of the distinct advantages is that, once the bus

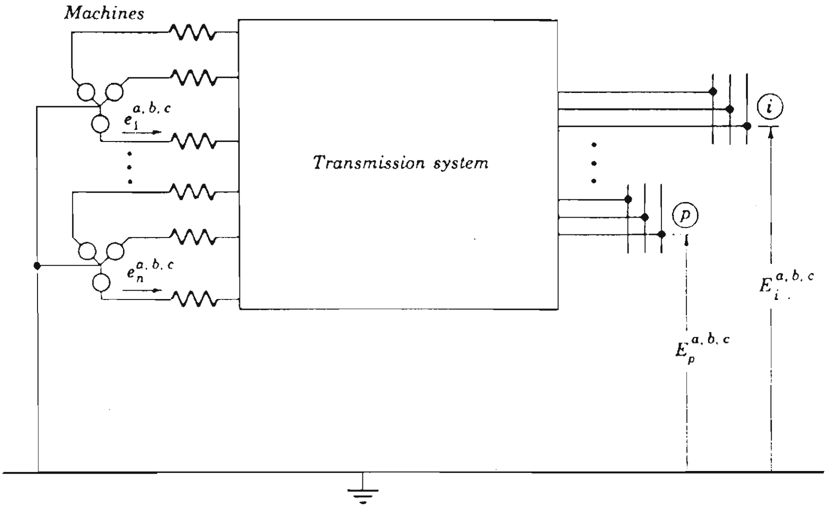


Fig. 6.2 Three-phase representation of a power system for short circuit studies.

impedance matrix is formed, the elements of this matrix can be used directly to calculate the currents and voltages associated with various types of faults and fault locations.

The representation of the system with a fault at bus p is shown in Fig. 6.3. In this representation, derived by means of Thevenin's theorem, the internal impedance is represented by the bus impedance matrix including machine reactances, and the open-circuited voltage is represented by the bus voltages prior to the fault.

The performance equation of the system during a fault is

$$\bar{E}^{a,b,c}_{BUS(F)} = \bar{E}^{a,b,c}_{BUS(0)} - Z^{a,b,c}_{BUS}\bar{I}^{a,b,c}_{BUS(F)} \tag{6.2.1}$$

The unknown voltage vector is

$$\bar{E}^{a,b,c}_{BUS(F)} = \begin{bmatrix} E^{a,b,c}_{1(F)} \\ \hline \cdots \\ \hline E^{a,b,c}_{p(F)} \\ \hline \cdots \\ \hline E^{a,b,c}_{n(F)} \end{bmatrix}$$

where the elements of $\bar{E}^{a,b,c}_{BUS(F)}$ are the three-phase voltage vectors $E^{a,b,c}_{i(F)}$ $\quad i = 1, 2, \ldots, n$

The known voltage vector prior to the fault is

$$\bar{E}^{a,b,c}_{BUS(0)} = \begin{array}{|c|}\hline E^{a,b,c}_{1(0)} \\ \hline \cdots \\ \hline E^{a,b,c}_{p(0)} \\ \hline \cdots \\ \hline E^{a,b,c}_{n(0)} \\ \hline \end{array}$$

The unknown bus current vector during a fault at bus p is

$$\bar{I}^{a,b,c}_{BUS(F)} = \begin{array}{|c|}\hline 0 \\ \hline \cdots \\ \hline 0 \\ \hline I^{a,b,c}_{p(F)} \\ \hline 0 \\ \hline \cdots \\ \hline 0 \\ \hline \end{array}$$

The three-phase bus impedance matrix is

$$Z^{a,b,c}_{BUS} = \begin{array}{|c|c|c|c|c|}\hline Z^{a,b,c}_{11} & \cdots & Z^{a,b,c}_{1p} & \cdots & Z^{a,b,c}_{1n} \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline Z^{a,b,c}_{p1} & \cdots & Z^{a,b,c}_{pp} & \cdots & Z^{a,b,c}_{pn} \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline Z^{a,b,c}_{n1} & \cdots & Z^{a,b,c}_{np} & \cdots & Z^{a,b,c}_{nn} \\ \hline \end{array}$$

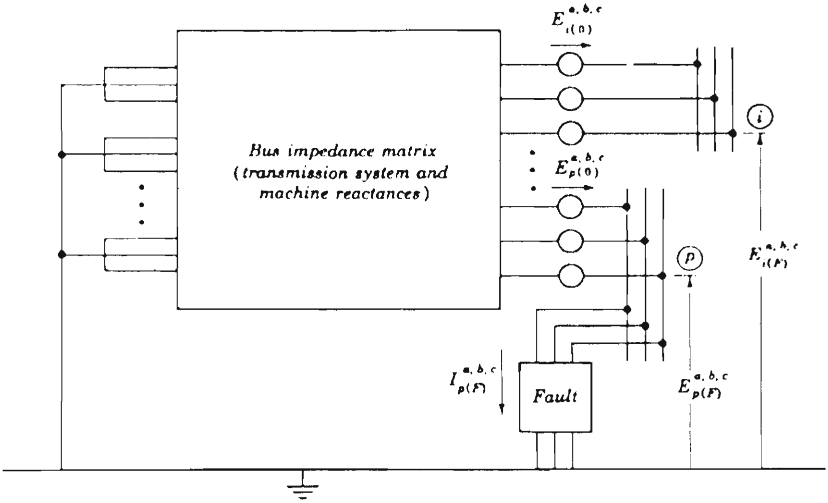


Fig. 6.3 Three-phase representation of a power system with a fault at bus p.

where the elements of $Z^{a,b,c}_{BUS}$ are matrices of dimension 3 × 3. Equation (6.2.1) can be written as follows:

$$\begin{aligned}
E^{a,b,c}_{1(F)} &= E^{a,b,c}_{1(0)} - Z^{a,b,c}_{1p} I^{a,b,c}_{p(F)} \\
E^{a,b,c}_{2(F)} &= E^{a,b,c}_{2(0)} - Z^{a,b,c}_{2p} I^{a,b,c}_{p(F)} \\
&\cdots\cdots\cdots \\
E^{a,b,c}_{p(F)} &= E^{a,b,c}_{p(0)} - Z^{a,b,c}_{pp} I^{a,b,c}_{p(F)} \\
&\cdots\cdots\cdots \\
E^{a,b,c}_{n(F)} &= E^{a,b,c}_{n(0)} - Z^{a,b,c}_{np} I^{a,b,c}_{p(F)}
\end{aligned} \tag{6.2.2}$$

The three-phase voltage vector at the faulted bus p is, from Fig. 6.3,

$$E^{a,b,c}_{p(F)} = Z^{a,b,c}_F I^{a,b,c}_{p(F)} \tag{6.2.3}$$

where $Z^{a,b,c}_F$ is the three-phase impedance matrix for the fault. The elements of this 3 × 3 matrix depend on the type of fault and fault impedance. Substituting from equation (6.2.3) for $E^{a,b,c}_{p(F)}$, the pth equation of (6.2.2) becomes

$$Z^{a,b,c}_F I^{a,b,c}_{p(F)} = E^{a,b,c}_{p(0)} - Z^{a,b,c}_{pp} I^{a,b,c}_{p(F)} \tag{6.2.4}$$

Solving equation (6.2.4) for $I^{a,b,c}_{p(F)}$ yields

$$I^{a,b,c}_{p(F)} = (Z^{a,b,c}_F + Z^{a,b,c}_{pp})^{-1} E^{a,b,c}_{p(0)} \tag{6.2.5}$$

Substituting for $I^{a,b,c}_{p(F)}$ in equation (6.2.3), the three-phase voltage at the faulted bus p is

$$E^{a,b,c}_{p(F)} = Z^{a,b,c}_{F}(Z^{a,b,c}_{F} + Z^{a,b,c}_{pp})^{-1}E^{a,b,c}_{p(0)} \tag{6.2.6}$$

Similarly, the three-phase voltages at buses other than p can be obtained by substituting for $I^{a,b,c}_{p(F)}$ from equation (6.2.5). Then

$$E^{a,b,c}_{i(F)} = E^{a,b,c}_{i(0)} - Z^{a,b,c}_{ip}(Z^{a,b,c}_{F} + Z^{a,b,c}_{pp})^{-1}E^{a,b,c}_{p(0)} \qquad i \neq p \tag{6.2.7}$$

When it is desirable to express the parameters of the fault circuit in the admittance form, the three-phase fault current at bus p is

$$I^{a,b,c}_{p(F)} = Y^{a,b,c}_{F}E^{a,b,c}_{p(F)} \tag{6.2.8}$$

where $Y^{a,b,c}_{F}$ is the three-phase admittance matrix for the fault. Substituting $I^{a,b,c}_{p(F)}$ from equation (6.2.8), the pth equation of (6.2.2) becomes

$$E^{a,b,c}_{p(F)} = E^{a,b,c}_{p(0)} - Z^{a,b,c}_{pp}Y^{a,b,c}_{F}E^{a,b,c}_{p(F)} \tag{6.2.9}$$

Solving equation (6.2.9) for $E^{a,b,c}_{p(F)}$ yields

$$E^{a,b,c}_{p(F)} = (U + Z^{a,b,c}_{pp}Y^{a,b,c}_{F})^{-1}E^{a,b,c}_{p(0)} \tag{6.2.10}$$

Substituting for $E^{a,b,c}_{p(F)}$ in equation (6.2.8), the three-phase current at the faulted bus p is

$$I^{a,b,c}_{p(F)} = Y^{a,b,c}_{F}(U + Z^{a,b,c}_{pp}Y^{a,b,c}_{F})^{-1}E^{a,b,c}_{p(0)} \tag{6.2.11}$$

Similarly, the three-phase voltages at buses other than p can be obtained by substituting for $I^{a,b,c}_{p(F)}$ from equation (6.2.11). Then

$$E^{a,b,c}_{i(F)} = E^{a,b,c}_{i(0)} - Z^{a,b,c}_{ip}Y^{a,b,c}_{F}(U + Z^{a,b,c}_{pp}Y^{a,b,c}_{F})^{-1}E^{a,b,c}_{p(0)} \qquad i \neq p \tag{6.2.12}$$

Fault currents flowing through the elements of the network can be calculated with the bus voltages obtained from equations (6.2.6) and (6.2.7) or from equations (6.2.10) and (6.2.12). These currents in terms of the voltages across the elements of the network are

$$\bar{i}^{a,b,c}_{(F)} = [y^{a,b,c}]\bar{v}^{a,b,c}_{(F)}$$

where the elements of the current vector are

$$i^{a,b,c}_{ij(F)} = \begin{array}{|c|} \hline i^{a}_{ij(F)} \\ \hline i^{b}_{ij(F)} \\ \hline i^{c}_{ij(F)} \\ \hline \end{array}$$

the elements of the voltage vector are

$$v^{a,b,c}_{ij(F)} = \begin{array}{|c|}\hline v^{a}_{ij(F)} \\ \hline v^{b}_{ij(F)} \\ \hline v^{c}_{ij(F)} \\ \hline \end{array}$$

and the elements of the primitive admittance matrix are

$$y^{a,b,c}_{ij,kl} = \begin{array}{|c|c|c|}\hline y^{aa}_{ij,kl} & y^{ab}_{ij,kl} & y^{ac}_{ij,kl} \\ \hline y^{ba}_{ij,kl} & y^{bb}_{ij,kl} & y^{bc}_{ij,kl} \\ \hline y^{ca}_{ij,kl} & y^{cb}_{ij,kl} & y^{cc}_{ij,kl} \\ \hline \end{array}$$

where $y^{bc}_{ij,kl}$ is the mutual admittance between phase b of network element i–j and phase c of network element k–l. The three-phase current in the network element i–j can be calculated from

$$\bar{i}^{a,b,c}_{ij(F)} = \bar{y}^{a,b,c}_{ij,\rho\sigma}\bar{v}^{a,b,c}_{\rho\sigma(F)} \tag{6.2.13}$$

where $\rho\sigma$ refers to the element i–j as well as to elements mutually coupled to i–j. Since

$$\bar{v}^{a,b,c}_{\rho\sigma(F)} = \bar{E}^{a,b,c}_{\rho(F)} - \bar{E}^{a,b,c}_{\sigma(F)}$$

then equation (6.2.13) becomes

$$\bar{i}^{a,b,c}_{ij(F)} = \bar{y}^{a,b,c}_{ij,\rho\sigma}(\bar{E}^{a,b,c}_{\rho(F)} - \bar{E}^{a,b,c}_{\sigma(F)}) \tag{6.2.14}$$

The formulas for fault currents and voltages derived in this section can be used for balanced and unbalanced three-phase short circuit studies.

6.3 Short circuit calculations for balanced three-phase network using Z_{BUS}

Transformation to symmetrical components

The formulas developed in the preceding section for calculation of fault currents and voltages can be simplified for a balanced three-phase network by using symmetrical components. The primitive impedance matrix for a stationary balanced three-phase element is

$$z_{pq}^{a,b,c} = \begin{bmatrix} z_{pq}^{s} & z_{pq}^{m} & z_{pq}^{m} \\ z_{pq}^{m} & z_{pq}^{s} & z_{pq}^{m} \\ z_{pq}^{m} & z_{pq}^{m} & z_{pq}^{s} \end{bmatrix}$$

This matrix can be diagonalized by the transformation $(T_s^*)^t z_{pq}^{a,b,c} T_s$ into

$$z_{pq}^{0,1,2} = \begin{bmatrix} z_{pq}^{(0)} & & \\ & z_{pq}^{(1)} & \\ & & z_{pq}^{(2)} \end{bmatrix}$$

where $z_{pq}^{(0)}$, $z_{pq}^{(1)}$, and $z_{pq}^{(2)}$ are the zero, positive, and negative sequence impedances, respectively. The positive and negative sequence impedances for a stationary balanced three-phase element are equal. In addition, it is generally accepted that positive and negative sequence impedances for rotating elements can be assumed equal for short circuit calculations.

In a similar manner, each $y_{ij,kl}^{a,b,c}$ in the primitive admittance matrix and each $Z_{ij}^{a,b,c}$ in the bus impedance matrix can be diagonalized by the transformation matrix T_s to obtain, respectively,

$$y_{ij,kl}^{0,1,2} = \begin{bmatrix} y_{ij,kl}^{(0)} & & \\ & y_{ij,kl}^{(1)} & \\ & & y_{ij,kl}^{(2)} \end{bmatrix} \quad \text{and} \quad Z_{ij}^{0,1,2} = \begin{bmatrix} Z_{ij}^{(0)} & & \\ & Z_{ij}^{(1)} & \\ & & Z_{ij}^{(2)} \end{bmatrix}$$

It is customary to assume that all bus voltages prior to the fault are equal in magnitude and phase angle. Assuming the magnitude of the line-to-ground voltage $E_{i(0)}$ equal to one per unit, then the ith bus voltage before the fault is

$$E_{i(0)}^{a,b,c} = \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix}$$

Transforming into symmetrical components, that is

$$E_{i(0)}^{0,1,2} = (T_s^*)^t E_{i(0)}^{a,b,c}$$

then

$$E_{i(0)}^{0,1,2} = \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix}$$

The fault impedance matrix $Z_F^{a,b,c}$ can be transformed by T_s into the matrix $Z_F^{0,1,2}$. The resulting matrix is diagonal if the fault is balanced. The fault impedance and admittance matrices in terms of three-phase and symmetrical components for various types of faults are given in Table 6.1.

Similarly, the equations for calculating fault currents and voltages can be written in terms of symmetrical components. The current at the faulted bus p is

$$I_{p(F)}^{0,1,2} = (Z_F^{0,1,2} + Z_{pp}^{0,1,2})^{-1} E_{p(0)}^{0,1,2} \tag{6.3.1}$$

or

$$I_{p(F)}^{0,1,2} = Y_F^{0,1,2}(U + Z_{pp}^{0,1,2} Y_F^{0,1,2})^{-1} E_{p(0)}^{0,1,2} \tag{6.3.2}$$

The voltage at the faulted bus p is

$$E_{p(F)}^{0,1,2} = Z_F^{0,1,2}(Z_F^{0,1,2} + Z_{pp}^{0,1,2})^{-1} E_{p(0)}^{0,1,2} \tag{6.3.3}$$

or

$$E_{p(F)}^{0,1,2} = (U + Z_{pp}^{0,1,2} Y_F^{0,1,2})^{-1} E_{p(0)}^{0,1,2} \tag{6.3.4}$$

The voltages at buses other than p are

$$E_{i(F)}^{0,1,2} = E_{i(0)}^{0,1,2} - Z_{ip}^{0,1,2}(Z_F^{0,1,2} + Z_{pp}^{0,1,2})^{-1} E_{p(0)}^{0,1,2} \tag{6.3.5}$$

or

$$E_{i(F)}^{0,1,2} = E_{i(0)}^{0,1,2} - Z_{ip}^{0,1,2} Y_F^{0,1,2}(U + Z_{pp}^{0,1,2} Y_F^{0,1,2})^{-1} E_{p(0)}^{0,1,2} \tag{6.3.6}$$

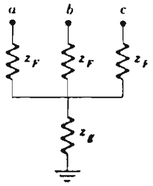
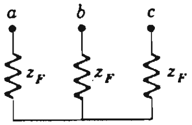
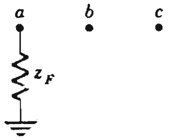
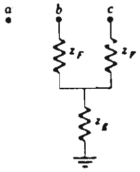
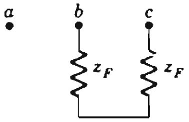
The fault current in the three-phase element i-j is

$$i_{ij(F)}^{0,1,2} = \bar{y}_{ij,\rho\sigma}^{0,1,2}(\bar{E}_{\rho(F)}^{0,1,2} - \bar{E}_{\sigma(F)}^{0,1,2}) \tag{6.3.7}$$

Three-phase-to-ground fault

Fault currents and voltages for a three-phase-to-ground fault can be obtained by substituting the corresponding fault impedance matrix, in

Table 6.1 Fault impedance and admittance matrices

Type of fault	Three-phase components	
	$Z_F^{a,b,c}$	$Y_F^{a,b,c}$
 Three-phase-to-ground	$\begin{bmatrix} z_F+z_g & z_g & z_g \\ z_g & z_F+z_g & z_g \\ z_g & z_g & z_F+z_g \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} y_0+2y_F & y_0-y_F & y_0-y_F \\ y_0-y_F & y_0+2y_F & y_0-y_F \\ y_0-y_F & y_0-y_F & y_0+2y_F \end{bmatrix}$ where $y_0 = \dfrac{1}{z_F+3z_g}$
 Three-phase	Not defined	$\frac{y_F}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$
 Line-to-ground	$\begin{bmatrix} z_F & 0 & 0 \\ 0 & \infty & 0 \\ 0 & 0 & \infty \end{bmatrix}$	$\begin{bmatrix} y_F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
 Line-to-line-to-ground	$\begin{bmatrix} \infty & 0 & 0 \\ 0 & z_F+z_g & z_g \\ 0 & z_g & z_F+z_g \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{z_F+z_g}{z_F^2+2z_Fz_g} & \dfrac{-z_g}{z_F^2+2z_Fz_g} \\ 0 & \dfrac{-z_g}{z_F^2+2z_Fz_g} & \dfrac{z_F+z_g}{z_F^2+2z_Fz_g} \end{bmatrix}$
 Line-to-line	Not defined	$\frac{y_F}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Symmetrical components	
$Z_F^{0,1,2}$	$Y_F^{0,1,2}$
$\begin{bmatrix} z_F + 3z_g & 0 & 0 \\ 0 & z_F & 0 \\ 0 & 0 & z_F \end{bmatrix}$	$\begin{bmatrix} y_0 & 0 & 0 \\ 0 & y_F & 0 \\ 0 & 0 & y_F \end{bmatrix}$ where $y_0 = \dfrac{1}{z_F + 3z_g}$
$\begin{bmatrix} \infty & 0 & 0 \\ 0 & z_F & 0 \\ 0 & 0 & z_F \end{bmatrix}$	$y_F \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Not defined	$\dfrac{y_F}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
Not defined	$\dfrac{1}{3(z_F^2 + 2z_F z_g)} \begin{bmatrix} 2z_F & -z_F & -z_F \\ -z_F & 2z_F + 3z_g & -(z_F + 3z_g) \\ -z_F & -(z_F + 3z_g) & 2z_F + 3z_g \end{bmatrix}$
Not defined	$\dfrac{y_F}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

terms of symmetrical components, into equations (6.3.1), (6.3.3), and (6.3.5). Both sides of the resulting equations can be premultiplied by T_s to obtain the corresponding formulas in terms of phase components.

The fault impedance matrix for a three-phase-to-ground fault is, from Table 6.1,

$$Z_F^{0,1,2} = \begin{bmatrix} z_F + 3z_g & & \\ & z_F & \\ & & z_F \end{bmatrix} \qquad (6.3.8)$$

The three-phase fault current and the bus voltages are obtained by substituting, from equation (6.3.8), for $Z_F^{0,1,2}$ in equations (6.3.1), (6.3.3), and (6.3.5). The current at the faulted bus p is

$$\begin{bmatrix} I_{p(F)}^{(0)} \\ I_{p(F)}^{(1)} \\ I_{p(F)}^{(2)} \end{bmatrix} = \left(\begin{bmatrix} z_F + 3z_g + Z_{pp}^{(0)} & & \\ & z_F + Z_{pp}^{(1)} & \\ & & z_F + Z_{pp}^{(1)} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} I_{p(F)}^{(0)} \\ I_{p(F)}^{(1)} \\ I_{p(F)}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{\sqrt{3}}{z_F + Z_{pp}^{(1)}} \\ 0 \end{bmatrix} \qquad (6.3.9)$$

The phase components of the fault current at bus p can be obtained by premultiplying both sides of equation (6.3.9) by T_s. These currents are

$$\begin{bmatrix} I_{p(F)}^{a} \\ I_{p(F)}^{b} \\ I_{p(F)}^{c} \end{bmatrix} = \frac{1}{z_F + Z_{pp}^{(1)}} \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix}$$

The voltage at the faulted bus p is

$$\begin{bmatrix} E^{(0)}_{p(F)} \\ E^{(1)}_{p(F)} \\ E^{(2)}_{p(F)} \end{bmatrix} = \begin{bmatrix} z_F + 3z_g & & \\ & z_F & \\ & & z_F \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{\sqrt{3}}{z_F + Z^{(1)}_{pp}} \\ 0 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} E^{(0)}_{p(F)} \\ E^{(1)}_{p(F)} \\ E^{(2)}_{p(F)} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{\sqrt{3}\, z_F}{z_F + Z^{(1)}_{pp}} \\ 0 \end{bmatrix}$$

The phase components of the fault voltage are

$$\begin{bmatrix} E^{a}_{p(F)} \\ E^{b}_{p(F)} \\ E^{c}_{p(F)} \end{bmatrix} = \frac{z_F}{z_F + Z^{(1)}_{pp}} \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix}$$

The voltages at buses other than p are

$$\begin{bmatrix} E^{(0)}_{i(F)} \\ E^{(1)}_{i(F)} \\ E^{(2)}_{i(F)} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} - \begin{bmatrix} Z^{(0)}_{ip} & & \\ & Z^{(1)}_{ip} & \\ & & Z^{(2)}_{ip} \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{\sqrt{3}}{z_F + Z^{(1)}_{pp}} \\ 0 \end{bmatrix}$$

which reduces to

$$\begin{array}{|c|}\hline E^{(0)}_{i(F)} \\ \hline E^{(1)}_{i(F)} \\ \hline E^{(2)}_{i(F)} \\ \hline \end{array} = \sqrt{3} \begin{array}{|c|}\hline 0 \\ \hline 1 - \dfrac{Z^{(1)}_{ip}}{z_p + Z^{(1)}_{pp}} \\ \hline 0 \\ \hline \end{array}$$

In phase components,

$$\begin{array}{|c|}\hline E^{a}_{i(F)} \\ \hline E^{b}_{i(F)} \\ \hline E^{c}_{i(F)} \\ \hline \end{array} = \left(1 - \frac{Z^{(1)}_{ip}}{z_F + Z^{(1)}_{pp}}\right) \begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array}$$

Table 6.2 Current and voltage formulas for three-phase-to-ground fault at bus p

Three-phase components	*Symmetrical components*
$I^{a,b,c}_{p(F)} = \dfrac{E_{p(0)}}{z_F + Z^{(1)}_{pp}} \begin{array}{c}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array}$	$I^{0,1,2}_{p(F)} = \dfrac{\sqrt{3}\, E_{p(0)}}{z_F + Z^{(1)}_{pp}} \begin{array}{c}\hline 0 \\ \hline 1 \\ \hline 0 \\ \hline \end{array}$
$E^{a,b,c}_{p(F)} = \dfrac{z_F E_{p(0)}}{z_F + Z^{(1)}_{pp}} \begin{array}{c}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array}$	$E^{0,1,2}_{p(F)} = \dfrac{\sqrt{3}\, z_F E_{p(0)}}{z_F + Z^{(1)}_{pp}} \begin{array}{c}\hline 0 \\ \hline 1 \\ \hline 0 \\ \hline \end{array}$
$E^{a,b,c}_{i(F)} = \left(E_{i(0)} - \dfrac{Z^{(1)}_{ip} E_{p(0)}}{z_F + Z^{(1)}_{pp}}\right) \begin{array}{c}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array}$ $i \neq p$	$E^{0,1,2}_{i(F)} = \sqrt{3} \left(E_{i(0)} - \dfrac{Z^{(1)}_{ip} E_{p(0)}}{z_F + Z^{(1)}_{pp}}\right) \begin{array}{c}\hline 0 \\ \hline 1 \\ \hline 0 \\ \hline \end{array}$ $i \neq p$

The formulas derived in this section are summarized in Table 6.2. The line-to-ground voltage was assumed to be one per unit in the derivations. The formulas in Table 6.2 include the term for the line-to-ground voltage which can be set at any desired per unit value.

The currents in the network elements during the fault can be calculated from equation (6.3.7). Since the zero and negative sequence bus voltages are zero for a three-phase fault and there is no mutual coupling in the positive sequence network, that is, $y^{(1)}_{ij,\rho\sigma} = 0$ except when $\rho\sigma = ij$, then equation (6.3.7) reduces to

$$\begin{array}{|c|}\hline i^{(0)}_{ij(F)} \\ \hline i^{(1)}_{ij(F)} \\ \hline i^{(2)}_{ij(F)} \\ \hline \end{array} = \begin{array}{|c|}\hline 0 \\ \hline y^{(1)}_{ij,ij}(E^{(1)}_{i(F)} - E^{(1)}_{j(F)}) \\ \hline 0 \\ \hline \end{array}$$

In phase components,

$$\begin{array}{|c|}\hline i^{a}_{ij(F)} \\ \hline i^{b}_{ij(F)} \\ \hline i^{c}_{ij(F)} \\ \hline \end{array} = \frac{1}{\sqrt{3}} y^{(1)}_{ij,ij}(E^{(1)}_{i(F)} - E^{(1)}_{j(F)}) \begin{array}{|c|}\hline 1 \\ \hline a^2 \\ \hline a \\ \hline \end{array}$$

Line-to-ground fault

The fault admittance matrix for a line-to-ground fault in phase a is, from Table 6.1,

$$Y^{0,1,2}_F = \frac{y_F}{3} \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \qquad (6.3.10)$$

The fault current and the bus voltages are obtained by substituting from equation (6.3.10) for $Y^{0,1,2}_F$ in equations (6.3.2), (6.3.4), and (6.3.6). The current at the faulted bus p is

$$\begin{bmatrix} I^{(0)}_{p(F)} \\ I^{(1)}_{p(F)} \\ I^{(2)}_{p(F)} \end{bmatrix} = \frac{y_F}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 + Z^{(0)}_{pp}\frac{y_F}{3} & Z^{(0)}_{pp}\frac{y_F}{3} & Z^{(0)}_{pp}\frac{y_F}{3} \\ Z^{(1)}_{pp}\frac{y_F}{3} & 1 + Z^{(1)}_{pp}\frac{y_F}{3} & Z^{(1)}_{pp}\frac{y_F}{3} \\ Z^{(1)}_{pp}\frac{y_F}{3} & Z^{(1)}_{pp}\frac{y_F}{3} & 1 + Z^{(1)}_{pp}\frac{y_F}{3} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} I^{(0)}_{p(F)} \\ I^{(1)}_{p(F)} \\ I^{(2)}_{p(F)} \end{bmatrix} = \frac{\sqrt{3}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{6.3.11}$$

The phase components of the fault current at bus p can be obtained by premultiplying both sides of equation (6.3.11) by T_s. These currents are

$$\begin{bmatrix} I^{a}_{p(F)} \\ I^{b}_{p(F)} \\ I^{c}_{p(F)} \end{bmatrix} = \begin{bmatrix} \dfrac{3}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F} \\ 0 \\ 0 \end{bmatrix}$$

The voltage at the faulted bus p is

$$\begin{bmatrix} E^{(0)}_{p(F)} \\ E^{(1)}_{p(F)} \\ E^{(2)}_{p(F)} \end{bmatrix} = \left(\begin{bmatrix} 1 + Z^{(0)}_{pp}\frac{y_F}{3} & Z^{(0)}_{pp}\frac{y_F}{3} & Z^{(0)}_{pp}\frac{y_F}{3} \\ Z^{(1)}_{pp}\frac{y_F}{3} & 1 + Z^{(1)}_{pp}\frac{y_F}{3} & Z^{(1)}_{pp}\frac{y_F}{3} \\ Z^{(1)}_{pp}\frac{y_F}{3} & Z^{(1)}_{pp}\frac{y_F}{3} & 1 + Z^{(1)}_{pp}\frac{y_F}{3} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} E^{(0)}_{p(F)} \\ E^{(1)}_{p(F)} \\ E^{(2)}_{p(F)} \end{bmatrix} = \frac{\sqrt{3}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F} \begin{bmatrix} -Z^{(0)}_{pp} \\ Z^{(0)}_{pp} + Z^{(1)}_{pp} + 3z_F \\ -Z^{(1)}_{pp} \end{bmatrix}$$

The phase components of the fault voltage are

$$\begin{bmatrix} E^{a}_{p(F)} \\ E^{b}_{p(F)} \\ E^{c}_{p(F)} \end{bmatrix} = \begin{bmatrix} \dfrac{3z_F}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F} \\ a^2 - \dfrac{Z^{(0)}_{pp} - Z^{(1)}_{pp}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F} \\ a - \dfrac{Z^{(0)}_{pp} - Z^{(1)}_{pp}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_P} \end{bmatrix}$$

The voltages at buses other than p are

$$\begin{bmatrix} E^{(0)}_{i(F)} \\ E^{(1)}_{i(F)} \\ E^{(2)}_{i(F)} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} - \begin{bmatrix} Z^{(0)}_{ip} & & \\ & Z^{(1)}_{ip} & \\ & & Z^{(1)}_{ip} \end{bmatrix} \frac{\sqrt{3}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} E^{(0)}_{i(F)} \\ E^{(1)}_{i(F)} \\ E^{(2)}_{i(F)} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} - \frac{\sqrt{3}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F} \begin{bmatrix} Z^{(0)}_{ip} \\ Z^{(1)}_{ip} \\ Z^{(1)}_{ip} \end{bmatrix}$$

In phase components,

$$\begin{bmatrix} E^{a}_{i(F)} \\ E^{b}_{i(F)} \\ E^{c}_{i(F)} \end{bmatrix} = \begin{bmatrix} 1 \\ a^2 \\ a \end{bmatrix} - \frac{1}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F} \begin{bmatrix} Z^{(0)}_{ip} + 2Z^{(1)}_{ip} \\ Z^{(0)}_{ip} - Z^{(1)}_{ip} \\ Z^{(0)}_{ip} - Z^{(1)}_{ip} \end{bmatrix}$$

Table 6.3 Current and voltage formulas for line-to-ground fault (phase a) at bus p

Three-phase components	Symmetrical components
$I^{a,b,c}_{p(F)} = \frac{3E_{p(0)}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\begin{bmatrix}1\\0\\0\end{bmatrix}$	$I^{0,1,2}_{p(F)} = \frac{\sqrt{3}\,E_{p(0)}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\begin{bmatrix}1\\1\\1\end{bmatrix}$
$E^{a,b,c}_{p(F)} = E_{p(0)}\begin{bmatrix}\frac{3z_F}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\\ a^2 - \frac{Z^{(0)}_{pp} - Z^{(1)}_{pp}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\\ a - \frac{Z^{(0)}_{pp} - Z^{(1)}_{pp}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\end{bmatrix}$	$E^{0,1,2}_{p(F)} = \frac{\sqrt{3}\,E_{p(0)}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\begin{bmatrix}-Z^{(0)}_{pp}\\ Z^{(0)}_{pp} + Z^{(1)}_{pp} + 3z_F\\ -Z^{(1)}_{pp}\end{bmatrix}$
$E^{a,b,c}_{i(F)} = E_{i(0)}\begin{bmatrix}1\\a^2\\a\end{bmatrix} - E_{p(0)}\begin{bmatrix}\frac{Z^{(0)}_{ip} + 2Z^{(1)}_{ip}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\\ \frac{Z^{(0)}_{ip} - Z^{(1)}_{ip}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\\ \frac{Z^{(0)}_{ip} - Z^{(1)}_{ip}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\end{bmatrix}$ $i \neq p$	$E^{0,1,2}_{i(F)} = E_{i(0)}\begin{bmatrix}0\\\sqrt{3}\\0\end{bmatrix} - \frac{\sqrt{3}\,E_{p(0)}}{Z^{(0)}_{pp} + 2Z^{(1)}_{pp} + 3z_F}\begin{bmatrix}Z^{(0)}_{ip}\\ Z^{(1)}_{ip}\\ Z^{(1)}_{ip}\end{bmatrix}$ $i \neq p$

The formulas derived in this section are summarized in Table 6.3. The line-to-ground voltage was assumed to be one per unit in the derivations. The formulas in Table 6.3 include the term for the line-to-ground voltage which can be set at any desired per unit value.

The currents in the network elements during the fault can be calculated from equation (6.3.7).

6.4 Example of short circuit calculations using Z_{BUS}

The method of calculating short circuit currents and voltages will be illustrated for the sample system shown in Fig. 6.4*a*. The oriented connected graph of this system is shown in Fig. 6.4*b*. This sample system is identical to the one used in Sec. 5.9.

Problem

a. Using symmetrical components, calculate the following for a three-phase fault at bus 4:

- i. Total fault current
- ii. Bus voltages during fault
- iii. Short circuit currents in lines connected to the faulted bus

b. Using symmetrical components, calculate the following for a line-to-ground fault at bus 4:

- i. Total fault current
- ii. Bus voltages during fault
- iii. Short circuit currents in lines connected to the faulted bus.

c. Determine the maximum three-phase short circuit current that circuit breaker *A* must interrupt for a fault on the line side of the breaker.

Solution

a. The bus impedance matrix in terms of sequence quantities must be determined to calculate three-phase and line-to-ground fault currents using symmetrical components. Table 5.2 shows the three-phase impedances of the network elements. The zero, positive, and negative sequence impedances of the network elements can be obtained by means of the transformation matrix T_s, that is,

$$z_{pq}^{0,1,2} = (T_s^*)^t z_{pq}^{a,b,c} T_s$$

Assuming the impedance matrices of the generators are symmetric and using the average value −0.0225 for the off-diagonal elements, the sequence impedances are shown in Table 6.4.

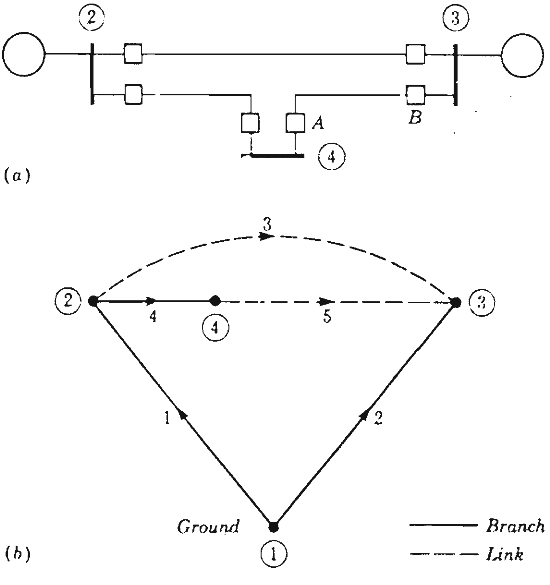


Fig. 6.4 Sample system for short circuit calculations. (a) Single line diagram of three-phase system; (b) oriented connected graph.

Table 6.4 Zero, positive, and negative sequence impedances for sample system

		Self	Mutual	
Element number	Bus code p–q	Impedance $z^{0,1,2}_{pq,pq}$	Bus code r–s	Impedance $z^{0,1,2}_{pq,rs}$
1	1–2	$\begin{array}{\|c\|c\|c\|}\hline 0.035 & & \\ \hline & 0.1025 & \\ \hline & & 0.1025 \\ \hline\end{array}$		
2	1–3	$\begin{array}{\|c\|c\|c\|}\hline 0.035 & & \\ \hline & 0.1025 & \\ \hline & & 0.1025 \\ \hline\end{array}$		
3	2–3	$\begin{array}{\|c\|c\|c\|}\hline 2.50 & & \\ \hline & 1.00 & \\ \hline & & 1.00 \\ \hline\end{array}$		
4	2–4	$\begin{array}{\|c\|c\|c\|}\hline 1.00 & & \\ \hline & 0.40 & \\ \hline & & 0.40 \\ \hline\end{array}$	2–3	$\begin{array}{\|c\|c\|c\|}\hline 0.60 & & \\ \hline & & \\ \hline & & \\ \hline\end{array}$
5	4–3	$\begin{array}{\|c\|c\|c\|}\hline 1.50 & & \\ \hline & 0.60 & \\ \hline & & 0.60 \\ \hline\end{array}$	2–3	$\begin{array}{\|c\|c\|c\|}\hline 0.90 & & \\ \hline & & \\ \hline & & \\ \hline\end{array}$

Since there is no coupling between the sequence impedances, the bus impedance matrix in terms of sequence quantities can be obtained by forming the positive, negative, and zero sequence bus impedance matrices independently. First, the positive sequence bus impedance matrix will be formed.

Step 1. Start with element 1, the branch from $p = 1$ to $q = 2$. The positive sequence bus impedance matrix for the partial network is

$$Z^{(1)}_{BUS} = \begin{array}{c|c|}
 & \textcircled{2} \\ \cline{2-2}
\textcircled{2} & 0.1025 \\ \cline{2-2}
\end{array}$$

Step 2. Add element 2, the branch from $p = 1$ to $q = 3$. Then,

$$Z^{(1)}_{BUS} = \begin{array}{c|c|c|}
 & \textcircled{2} & \textcircled{3} \\ \cline{2-3}
\textcircled{2} & 0.1025 & \\ \cline{2-3}
\textcircled{3} & & 0.1025 \\ \cline{2-3}
\end{array}$$

Step 3. Add element 4, the branch from $p = 2$ to $q = 4$. Thus,

$Z_{24} = Z_{42} = Z_{22}$
$Z_{34} = Z_{43} = 0$
$Z_{44} = Z_{24} + z_{24,24}$

and

$$Z^{(1)}_{BUS} = \begin{array}{c|c|c|c|}
 & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \cline{2-4}
\textcircled{2} & 0.1025 & & 0.1025 \\ \cline{2-4}
\textcircled{3} & & 0.1025 & \\ \cline{2-4}
\textcircled{4} & 0.1025 & & 0.5025 \\ \cline{2-4}
\end{array}$$

Step 4. Add element 5, the link from $p = 4$ to $q = 3$. The elements of the row and column corresponding to the fictitious node l are

$Z_{l2} = Z_{2l} = Z_{42} - Z_{32}$
$Z_{l3} = Z_{3l} = Z_{43} - Z_{33}$
$Z_{l4} = Z_{4l} = Z_{44} - Z_{34}$
$Z_{ll} = Z_{4l} - Z_{3l} + z_{43.43}$

and the augmented matrix is

$$\begin{array}{c|c|c|c|c|}
 & \textcircled{2} & \textcircled{3} & \textcircled{4} & l \\ \cline{2-5}
\textcircled{2} & 0.1025 & & 0.1025 & 0.1025 \\ \cline{2-5}
\textcircled{3} & & 0.1025 & & -0.1025 \\ \cline{2-5}
\textcircled{4} & 0.1025 & & 0.5025 & 0.5025 \\ \cline{2-5}
l & 0.1025 & -0.1025 & 0.5025 & 1.2050 \\ \cline{2-5}
\end{array}$$

To eliminate the lth row and column the elements of the augmented matrix are modified as follows:

$$Z'_{22} = Z_{22} - Z_{2l}Z_{ll}^{-1}Z_{l2}$$
$$Z'_{33} = Z_{33} - Z_{3l}Z_{ll}^{-1}Z_{l3}$$
$$Z'_{44} = Z_{44} - Z_{4l}Z_{ll}^{-1}Z_{l4}$$
$$Z'_{23} = Z'_{32} = Z_{23} - Z_{2l}Z_{ll}^{-1}Z_{l3}$$
$$Z'_{24} = Z'_{42} = Z_{24} - Z_{2l}Z_{ll}^{-1}Z_{l4}$$
$$Z'_{34} = Z'_{43} = Z_{34} - Z_{3l}Z_{ll}^{-1}Z_{l4}$$

Thus,

$$Z^{(1)}_{BUS} = \begin{array}{c|c|c|c|} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline \textcircled{2} & 0.0938 & 0.0087 & 0.0598 \\ \hline \textcircled{3} & 0.0087 & 0.0938 & 0.0427 \\ \hline \textcircled{4} & 0.0598 & 0.0427 & 0.2930 \\ \hline \end{array}$$

Step 5. Add element 3, the link from $p = 2$ to $q = 3$. As in the previous step,

$$Z_{l2} = Z_{2l} = Z_{22} - Z_{32}$$
$$Z_{l3} = Z_{3l} = Z_{23} - Z_{33}$$
$$Z_{l4} = Z_{4l} = Z_{24} - Z_{34}$$
$$Z_{ll} = Z_{2l} - Z_{3l} + z_{23,23}$$

$$\begin{array}{c|c|c|c|c|} & \textcircled{2} & \textcircled{3} & \textcircled{4} & l \\ \hline \textcircled{2} & 0.0938 & 0.0087 & 0.0598 & 0.0851 \\ \hline \textcircled{3} & 0.0087 & 0.0938 & 0.0427 & -0.0851 \\ \hline \textcircled{4} & 0.0598 & 0.0427 & 0.2930 & 0.0171 \\ \hline l & 0.0851 & -0.0851 & 0.0171 & 1.1702 \\ \hline \end{array}$$

Eliminating the lth row and column, the final positive sequence bus impedance matrix is

$$Z^{(1)}_{BUS} = \begin{array}{c|c|c|c|} & \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \hline \textcircled{2} & 0.0876 & 0.0149 & 0.0586 \\ \hline \textcircled{3} & 0.0149 & 0.0876 & 0.0439 \\ \hline \textcircled{4} & 0.0586 & 0.0439 & 0.2928 \\ \hline \end{array}$$

Since positive and negative primitive sequence impedances are equal, the positive and negative sequence bus impedance matrices are equal.

The procedure for forming the zero sequence bus impedance matrix is identical for the first four steps. The zero sequence bus impedance matrix of the partial network, before adding element 3, is

$$Z_{BUS}^{(0)} = \begin{array}{c|c|c|c|} & ② & ③ & ④ \\ \hline ② & 0.0345 & 0.0005 & 0.0209 \\ \hline ③ & 0.0005 & 0.0345 & 0.0141 \\ \hline ④ & 0.0209 & 0.0141 & 0.6182 \\ \hline \end{array}$$

Step 5. Add element 3, the link from $p = 2$ to $q = 3$, which is coupled with the elements 4 and 5. The elements of the row and column corresponding to the fictitious node l are

$$Z_{l2} = Z_{22} - Z_{32} + \frac{y_{23,24}(Z_{22} - Z_{42}) + y_{23,43}(Z_{42} - Z_{32})}{y_{23,23}}$$

$$Z_{l3} = Z_{23} - Z_{33} + \frac{y_{23,24}(Z_{23} - Z_{43}) + y_{23,43}(Z_{43} - Z_{33})}{y_{23,23}}$$

$$Z_{l4} = Z_{24} - Z_{34} + \frac{y_{23,24}(Z_{24} - Z_{44}) + y_{23,43}(Z_{44} - Z_{34})}{y_{23,23}}$$

$$Z_{ll} = Z_{2l} - Z_{3l} + \frac{1 + y_{23,24}(Z_{2l} - Z_{4l}) + y_{23,43}(Z_{4l} - Z_{3l})}{y_{23,23}}$$

The zero sequence primitive impedance matrix is

	1–2	1–3	2–3	2–4	4–3
1–2	0.035				
1–3		0.035			
2–3			2.500	0.600	0.900
2–4			0.600	1.000	
4–3			0.900		1.500